社会的選択理論の数学的研究I

(Mathematical Studies of Social Choice Theory I)

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『代数的トポロジーと社会的選択理論（後編）』改題

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まえがき

本稿は科学研究費補助金（基盤研究 C（一般）課題番号 16530128，平成 16〜19 年度）および平成 18, 19 年度私立大学等経常費補助金特別補助高度化推進特別経費大学院重点特別経費を受けて行った研究をまとめたものである。拙い英語で書いた論文をすべて少しはましな日本語に直そうかと思ったが，あまりに面倒かつ非生産的なので Chapter 1 の途中でやめてしまった。

Chapter 1 から Chapter 5 までは代数的トポロジー (Algebraic Topology) の手法を用いた社会的選択理論のいくつかの定理の証明を取り上げている。Chapter 6 から Chapter 8 では HEX ゲームと呼ばれるゲームに必ず勝者が存在するという定理と社会的選択理論の定理との関係を考察している。Chapter 9 から Chapter 12 までは無限人口のもとでの社会的選択ルールの計算可能性について，構成的数学 (Constructive Mathematics) や計算可能解析学 (Computable Calculus) などいくつかの観点から検討した研究であり，最後の Chapter 13 は構成的数学の概念を均衡理論に応用した研究である

もともとは『代数的トポロジーと社会的選択理論（後編)』というタイトルで作る予定であったが研究の方向性が変わりタイトルも変わってしまった。したがって『代数的トポロジーと社会的選択理論』は前編のみである。

各章の主な内容は以下のとおり。

Chapter 1：人数が 2 人，選択肢が 3 つのケースにおけるアローの不可能性定理が 2 次元球（円）についてのブラウワーの不動点定理と同値であることを代数的トポロジーの手法（ホモロジー群，写像度）で証明した。

Chapter 2：人々の選好が無差別な関係を含む場合のアローの不可能性定理についての代数的トポロジーを用いた証明。

Chapter 3：Eliaz(2004) による社会的選択理論に関する統一的な定理に代数的トポロジーの手法を用いた証明を与えた。

Chapter 4：アローの不可能性定理とアマルティア・センによるいわゆるリベラルパラドックスを代数的トポロジーの手法で分析し，これらがある抽象的な定理の特殊ケースとして表せることを示した。

Chapter 5：パレート原理を仮定しないでアローの不可能性定理を一般化したウィルソンの（不可能性）定理の代数的トポロジーによる証明。

Chapter 6：HeX ゲームと呼ばれるゲームに必ず勝者が存在するという定理とアローの不可能性定理との関係を人々の選好が無差別な関係を含まない強い選好である場合について検討した。

Chapter 7：HeX ゲームに必ず勝者が存在するという定理と Duggan and Schwartz(2000) による社会的選択対応に関する不可能性定理との関係を考察した。

Chapter 8：HeX ゲームに必ず勝者が存在するという定理とアローの不可能性定理との関係を人々の選好が無差別な関係を含む弱い選好である場合について検討した。

Chapter 9：Type 2 computability という概念を用いて無限人口社会における社会的選択関数の計算可能性について検討し，有限人口社会におけるギバード・サタースウェイトが主張するように独裁者が存在する場合には社会的選択関数は計算可能であるが，独裁者が存在しない場合には計算可能ではないことを示した。

Chapter 10：無限人口社会における（アロー的な）社会的厚生関数について構成的数学 (constructive mathematics) の観点から検討し，「社会的厚生関数は独裁者を持つかまたは独裁者を持たない」という言明は構成的数学における LPO(Limited principle of omniscience) と同値であるということを示した。

Chapter 11：無限人口社会における社会的選択関数について構成的数学 (constructive mathematics) の

観点から検討し，「社会的選択関数は独裁者を持つかまたは独裁者を持たない」という言明は構成的数学における LPO(Limited principle of omniscience) と同値であるということを示した。

Chapter 12：無限人口社会における（アロー的な）社会的厚生関数が独裁者を持つか持たないかは決定不可能であり，そのことは Turing 機械の停止問題 (halting problem) が決定不可能であるということと同値であることを示した。

Chapter 13：一般均衡理論における宇沢の同値定理（競争均衡の存在定理がブラウワーの不動点定理と同値である）を構成的数学の観点から検討し，そこで仮定される競争均衡価格の存在が構成的数学における LLPO(Lesser limited principle of omniscience) と同値であるということを示した。

2008 年 2 月 20 日　田中 靖人

List of First Publications

Chapter 1 "On the equivalence of the Arrow impossibility theorem and the Brouwer fixed point theorem", *Applied Mathematics and Computation*, Vol. 172, No. 2, pp. 1303-1314, 2006, Elsevier.

Chapter 2 "A topological approach to the Arrow impossibility theorem when individual preferences are weak orders", *Applied Mathematics and Computation*, Vol. 174, No. 2, pp. 961-981, 2006, Elsevier.

Chapter 3 "A topological proof of Eliaz's unified theorem of social choice theory", *Applied Mathematics and Computation*, Vol. 176, No. 1, pp. 83-90, 2006, Elsevier.

Chapter 4 "On the topological equivalence of the Arrow impossibility theorem and Amartya Sen's liberal paradox", *Applied Mathematics and Computation*, Vol. 181, No. 2, pp. 1490-1498, 2006, Elsevier.

Chapter 5 "A topological approach to Wilson's impossibility theorem", *Journal of Mathematical Economics*, Vol. 43, No. 2, pp. 184-191, 2007, Elsevier.

Chapter 6 "Equivalence of the HEX game theorem and the Arrow impossibility theorem", *Applied Mathematics and Computation*, Vol. 186, No. 1, pp. 509-515, 2007, Elsevier.

Chapter 7 "On the equivalence of the HEX game theorem and the Duggan-Schwartz theorem for strategy-proof social choice correspondences", *Applied Mathematics and Computation*, Vol. 188, No. 1, pp. 303-313, 2007, Elsevier.

Chapter 8 "The HEX game theorem and the Arrow impossibility theorem: the case of weak orders", *Metroeconomica*, forthcoming, Blackwell.

Chapter 9 "Type two computability of social choice functions and the Gibbard-Satterthwaite theorem in an infinite society", *Applied Mathematics and Computation*, Vol. 192, No. 1, pp. 168-174, 2007, Elsevier.

Chapter 10 "The Arrow impossibility theorem of social choice theory in an infinite society and LPO (Limited principle of omniscience)", *Applied Mathematics E-Notes*, forthcoming, National Tsing Hua University (Taiwan).

Chapter 11 "The Gibbard-Satterthwaite theorem of social choice theory in an infinite society and LPO (Limited principle of omniscience)", *Applied Mathematics and Computation*, Vol. 193, No. 2, pp. 475-481, 2007, Elsevier.

Chapter 12 "On the computability of binary social choice rules in an infinite society and the halting problem", *Applied Mathematics and Computation*, forthcoming, Elsevier.

Chapter 13 "Undecidability of Uzawa equivalence theorem and LLPO (Lesser limited principle of omniscience)", *Applied Mathematics and Computation*, forthcoming, Elsevier.

Contents

Chapter 1

On the equivalence of the Arrow impossibility theorem and the Brouwer fixed point theorem

We will show that in the case where there are two individuals and three alternatives (or under the assumption of free-triple property) the Arrow impossibility theorem (Arrow (1963)) for social welfare functions that there exists no social welfare function which satisfies transitivity, Pareto principle, independence of irrelevant alternatives, and has no dictator is equivalent to the Brouwer fixed point theorem on a 2-dimensional ball (circle). Our study is an application of ideas by Chichilnisky (1979) to a discrete social choice problem, and also it is in line with the work by Baryshnikov (1993). But tools and techniques of algebraic topology which we will use are more elementary than those in Baryshnikov (1993)[*1].

1.1 Introduction

Topological approaches to social choice problems have been initiated by Chichilnisky (1980). In her model a space of alternatives is a subset of Euclidean space, and individual preferences over this set are represented by normalized gradient fields. Her main result is an impossibility theorem that there exists no *continuous* social choice rule which satisfies *unanimity* and *anonymity*. This approach has been further developed by Chichilnisky (1979), (1982), Koshevoy (1997), Lauwers (2004), and so on. In particular, by Chichilnisky (1979) the equivalence of her impossibility result and the Brouwer fixed point theorem in the case where there are two individuals and the choice space is a subset of 2-dimensional Euclidian space has been shown. On the other hand, Baryshnikov (1993) and (1997) have presented a topological approach to Arrow's general possibility theorem, which is usually called the *Arrow impossibility theorem* (Arrow (1963)), in a discrete framework of social choice[*2].

We will examine the relationship between the Arrow impossibility theorem for social welfare functions that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of

[*1] This chapter is based on my paper of the same title published in *Applied Mathematics and Computation*, Vol. 172, No. 2, pp. 1303-1314, 2006, Elsevier.

[*2] About surveys and basic results of topological social choice theories, see Mehta (1997) and Lauwers (2000).

irrelevant alternatives and has no dictator*3, and the Brouwer fixed point theorem on a 2-dimensional ball in the case of two individuals and three alternatives (or under the assumption of free-triple property)*4. Our study is an application of ideas by Chichilnisky (1979) to a discrete social choice problem, and also it is in line with the work by Baryshnikov (1993). But tools and techniques of algebraic topology which we will use are more elementary than those used in Baryshnikov (1993). He used an advanced concept of algebraic topology, *nerve of a covering*. It is not contained in most elementary textbooks of algebraic topology, and is difficult of access for most economists. Our main tools are homology groups of simplicial complexes. Of course, the Brouwer fixed point theorem is a theorem about continuous functions. We will consider a method to obtain a continuous function from a discrete social choice rule. Mainly we will show the following results.

1. The Brouwer fixed point theorem is equivalent to the result that the restriction to an $n-1$-dimensional sphere S^{n-1} of a continuous function from an n-dimensional ball D^n to S^{n-1} is homotopic to a constant mapping.
2. The restriction of a continuous function obtained from a social welfare function which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator to a subset of the set of profiles of individual preferences, which is homeomorphic to a 2-dimensional ball (or circle) and the subset is homeomorphic to a 1-dimensional sphere (or circumference), is not homotopic to a constant mapping. It implies that the non-existence of social welfare function which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator is equivalent to the Brouwer fixed point theorem on a 2-dimensional ball.

In the next section we present the model of this chapter, and consider the homology groups of simplicial complexes which represent the set of individual preferences and the set of the social preference. In Section 1.3 we will show a result about the Brouwer fixed point theorem and homotopy of continuous functions. In Section 1.4 we will prove the main results.

1.2 The model

There are two individuals, A and B, and three alternatives of a social, economic or political problem, x_1, x_2 and x_3 (or we assume free-triple property). Individual preferences about these alternatives are not restricted. We assume that individual preferences for these alternatives are linear, that is, their preferences are always strict, and they are never indifferent about any pair of alternatives. Individual preferences must be complete and transitive. A social choice rule which we will consider is a rule which determines a preference of the society about x_1, x_2 and x_3 corresponding to a combination of preferences of two individuals. Transitive social choice rule is called a *social welfare function*. We require that social welfare functions satisfy Pareto principle and independence of irrelevant alternatives as well as transitivity. The means of the latter two conditions are as follows.

Pareto principle If all individuals prefer an alternative x_i to another alternative x_j, then the society must prefer x_i to x_j.

*3 Dictator is an individual whose (strict) preference always coincides with the social preference.

*4 Under the assumption of free-triple property, for each combination of three alternatives individual preferences are not restricted.

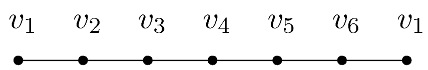

Figure 1: R

Independence of irrelevant alternatives The social preference about any pair of two alternatives x_i and x_j is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x_i and x_j.

The Arrow impossibility theorem states that there exists a dictator for any social welfare function which satisfies transitivity, Pareto principle and independence of irrelevant alternatives, or in other words there exists no social welfare function which satisfies these conditions and has no dictator. Dictator is an individual whose (strict) preference always coincides with the social preference.

From the set of individual preferences we draw a diagram by the following procedures.

1. When an individual prefers x_1 to x_2 to x_3, such a preference is denoted by (123), and corresponding to this preference we define a vertex v_1. Similarly, when an individual prefers x_1 to x_3 to x_2, such a preference is denoted by (132), and we define a vertex v_2. By similar procedures the following vertices are defined.

$$v_1 = (123),\ v_2 = (132),\ v_3 = (312),\ v_4 = (321),\ v_5 = (231),\ v_6 = (213)$$

   For example, $v_6 = (213)$ denotes a preference of an individual such that he prefers x_2 to x_1 to x_3.
2. These six vertices are plotted on a line segment in this order, locate v_1 at both end points, and connect the vertices.

Denote this diagram by R, and call $v_1, v_2, \cdots, v_6$ the vertices of R. It is depicted in Figure 1.

Two v_1's at both end points of R are not distinguished. The set of individual preferences is represented by R, and the set of combinations of the preferences of two individuals is represented by the product space $R \times R$. These combinations of individual preferences are called *preference profiles*. $R \times R$ is depicted as a square in Figure 2. The preference of individual B is represented from bottom up, not from left to right. Individual preferences are denoted by $p_A = v_1$, $p_B = v_2$ and so on, and preference profiles are denoted by $\mathbf{p} = (p_A, p_B) = (v_1, v_3)$, and so on.

The social preference is represented by a circumference depicted in Figure 3. We call this circumference S^1. The vertices of S^1 are denoted by $w_1, w_2, \cdots, w_6$. These vertices mean the following social preferences[*5].

1. w_1: The society prefers x_1 to x_2, x_2 to x_3.
2. w_2: The society prefers x_1 to x_3, x_3 to x_2.
3. w_3: The society prefers x_3 to x_1, x_1 to x_2.
4. w_4: The society prefers x_3 to x_2, x_2 to x_1.

[*5] From Lemma 1 of Baryshnikov (1993) we know that if individual preferences are strict orders, then the social preference is also a strict order under transitivity, Pareto principle and independence of irrelevant alternatives.

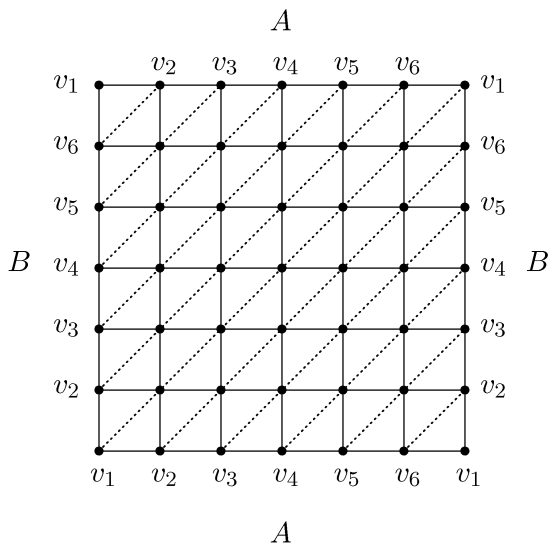

Figure 2: $R \times R$

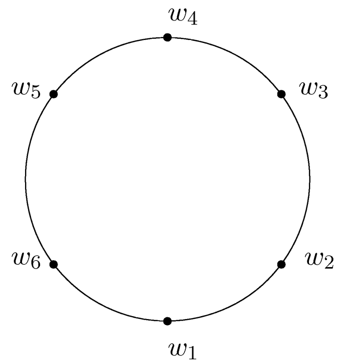

Figure 3: S^1

5. w_5: The society prefers x_2 to x_3, x_3 to x_1.
6. w_6: The society prefers x_2 to x_1, x_1 to x_3.

The 1-dimensional homology group (with integer coefficients) of S^1 is isomorphic to the group of integers $\mathbb{Z}$, that is, we have $H_1(S^1) \cong \mathbb{Z}$.

A social welfare function F is defined as a function from the vertices of $R \times R$ to the vertices of S^1. Let us consider a method to obtain a continuous function from a social welfare function defined on the vertices of $R \times R$. For example, for points included in a small triangle which consists of (v_1, v_3), (v_2, v_3) and (v_2, v_4) we define

$$F(\alpha(v_1, v_3) + \beta(v_2, v_3) + \gamma(v_2, v_4)) = \alpha F(v_1, v_3) + \beta F(v_2, v_3) + \gamma F(v_2, v_4)$$

where $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$ and $0 \leqq \gamma \leqq 1$, $\alpha + \beta + \gamma = 1$. Then, we can obtain a continuous function for the points in this triangle. By similar ways this continuous function is extended to the entire $R \times R$, and we obtain a continuous function for all points in $R \times R$ from a discrete social welfare function on the vertices of $R \times R$. Denote this continuous function by $F: R \times R \longrightarrow S^1$.

Let us see that this continuous function is well defined for the entire $R \times R$. By independence of irrelevant alternatives, for example, if $F(v_1, v_3) = w_1$, we must have $F(v_2, v_3) = w_1$ or $F(v_2, v_3) = w_2$. As this example shows, preferences represented by adjacent two vertices of $R \times R$ are identical about two pairs of alternatives. When the preference of one of two individuals changes, the social preference does not change, or it changes to one of adjacent vertices. Therefore, F is a *simplicial mapping*. If the preferences of two individuals change, the social preference moves at most two vertices clockwise or counter-clockwise on S^1, and hence the social preference does not change to the antipodal point or across the antipodal point on S^1. Thus, $\alpha F(v_1, v_3) + \beta F(v_2, v_3) + \gamma F(v_2, v_4)$ is well defined. Other cases are similar. Since F defined on the vertices of $R \times R$ is a simplicial mapping, we can define the homomorphism of homology groups induced by F. It is denoted by F_*.

Now we consider the following set Δ of vertices of $R \times R$.

$$\Delta = \{(v_1, v_1), (v_2, v_2), (v_3, v_3), (v_4, v_4), (v_5, v_5), (v_6, v_6), (v_1, v_1)\}$$

The diagram obtained by connecting these vertices is also denoted by Δ. It is homeomorphic to R. Preference profiles of two individuals when the preference of individual B is fixed at v_1, and preference profiles when the preference of individual A is fixed at v_1 are denoted, respectively, by $A = \{(p_A, p_B) : p_B = v_1\}$ and $B = \{(p_A, p_B) : p_A = v_1\}$. The diagrams obtained by connecting vertices of A, and similarly obtained from B are also denoted, respectively, by A and B. They are also homeomorphic to R. The union of these three sets $\Delta \cup A \cup B$ is depicted as the boundary ∂T_1 of the triangle T_1 in Figure 4. $\Delta \cup A \cup B$ is homeomorphic to the circumference S^1. The vertices at four corners of the square depicted in Figure 4 represent the same profile (v_1, v_1). The value of F for them are equal. The 1-dimensional homology group of $\Delta \cup A \cup B$ isomorphic to $\mathbb{Z}$, that is, $H_1(\Delta \cup A \cup B) \cong \mathbb{Z}$.

1.3 The Brouwer fixed point theorem

In this section we show the following theorem about homotopy and the degree of mapping of a continuous function on an $n-1$-dimensional sphere.

■Note Let F be a function from $n-1$-dimensional sphere S^{n-1} to itself, and F_* be the homomorphism of homology groups induced by F,

$$F_* : H_{n-1}(S^{n-1}) \longrightarrow H_{n-1}(S^{n-1})$$

$H_{n-1}(S^{n-1})$ is the $n-1$-dimensional homology group of S^{n-1}. Then, the degree of mapping of F, which is denoted by d_F, is defined as an integer which satisfies

$$F_*(h) = d_F h \text{ for } h \in H_{n-1}(S^{n-1})$$

Theorem 1.1 The following two results are equivalent.

1. If there exists a continuous function from an n-dimensional ball D^n to an $n-1$-dimensional sphere S^{n-1} ($n \geqq 2$), $F: D^n \longrightarrow S^{n-1}$, then the following function, which is obtained by restricting F to

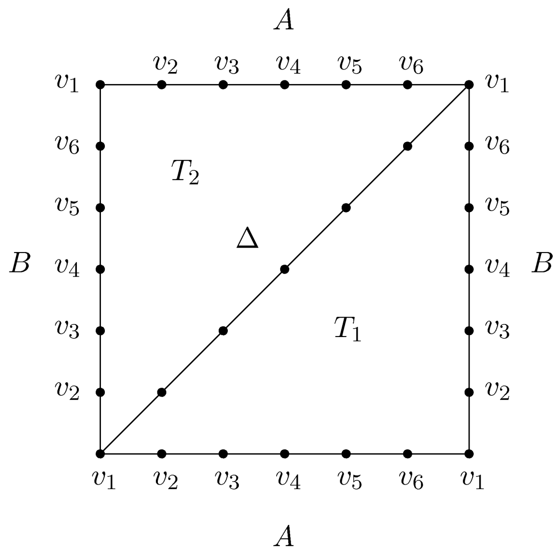

Figure 4: $\Delta \cup A \cup B$ and $R \times R$

the boundary S^{n-1} of D^n,

$$F|_{S^{n-1}} : S^{n-1} \longrightarrow S^{n-1}$$

is homotopic to a constant mapping. Since the degree of mapping of a constant mapping is zero, the degree of mapping of $F|_{S^{n-1}}$ is zero.

2. **(The Brouwer fixed point theorem)** Any continuous function from D^n to D^n ($n \geqq 2$), $G : D^n \longrightarrow D^n$, has a fixed point.

Proof. (1) $\longrightarrow$ (2)

Assume that G has no fixed point. Since we always have $v \neq G(v)$ at any point v in D^n, there is a half line starting $G(v)$ across v[*6]. Let $F(v)$ be the intersection point of this half line and the boundary of D^n, which is S^{n-1}. Then, we obtain the following continuous function from D^n to S^{n-1}.

$$F : D^n \longrightarrow S^{n-1}$$

In particular, we have $F(v) = v$ for $v \in S^{n-1}$. Therefore, $F|_{S^{n-1}}$ is an identity mapping. But, because an identity mapping on S^{n-1} is not homotopic to any constant mapping, it is a contradiction.

(2) $\longrightarrow$ (1)

We show that if there exists a continuous function F from D^n to S^{n-1}, (1) of this theorem is correct whether a continuous function G from D^n to D^n has a fixed point or not. Define $f_t(v) = F[(1-t)v](0 \leqq t \leqq 1)$ for any point v of S^{n-1}. Then, we get a continuous function $f_t : S^{n-1} \longrightarrow S^{n-1}$. $(1-t)v$ is a point which divide $t : 1-t$ a line segment between v and the center of D^n, and it is transferred by F to a point on S^{n-1}. We have $f_0 = F|_{S^{n-1}}$, and $f_1 = F(0)$ is a constant mapping whose image is a point $F(0)$. Since f_t is continuous with respect to t, it is a homotopy from $F|_{S^{n-1}}$ to a constant mapping, and the degree of mapping of $F|_{S^{n-1}}$ is zero.

[*6] If v is a fixed point, $G(v)$ and v coincide, and hence there does not exist such a half line.

□

An implication of this theorem is as follows.

Corollary 1.1 If there exists a function from D^n to S^{n-1}, $F: D^n \longrightarrow S^{n-1}$, and its restriction to S^{n-1}, $F|_{S^{n-1}}: S^{n-1} \longrightarrow S^{n-1}$, is not homotopic to a constant mapping, F can not be continuous.

In relation to a social welfare function on $R \times R$, if there exists a function F defined on the vertices of $R \times R$, we can obtain a continuous function on the entire $R \times R$ from F by the way explained above. Then, there exists a continuous function defined on T_1. Since T_1 is homeomorphic to D^2 (2-dimensional ball), and $\Delta \cup A \cup B$ is homeomorphic to S^1 (1-dimensional sphere), the restriction of F to $\Delta \cup A \cup B$, $F|_{\Delta \cup A \cup B}$, must be homotopic to a constant mapping. If, when we require that transitivity, Pareto principle, independence of irrelevant alternatives and the non-existence of dictator are satisfied by a social welfare function defined on the vertices of $R \times R$, the restriction of this function to $\Delta \cup A \cup B$ is not homotopic to a constant mapping, then there does not exist such a social welfare function in the first place.

1.4 The main results

From the preliminary analyses in the previous sections we can show the following lemma.

Lemma 1.1 Suppose that there exists a social welfare function $F: R \times R \longrightarrow S^1$ which satisfies transitivity, Pareto principle and independence of irrelevant alternatives. If F has no dictator, then the degree of mapping of $F|_{\Delta \cup A \cup B}$ is not zero, and hence it is not homotopic to a constant mapping.

Proof. By Pareto principle the vertices of Δ correspond to the vertices of S^1 as follows.

$$(v_1, v_1) \longrightarrow w_1,\ (v_2, v_2) \longrightarrow w_2,\ (v_3, v_3) \longrightarrow w_3$$

$$(v_4, v_4) \longrightarrow w_4,\ (v_5, v_5) \longrightarrow w_5,\ (v_6, v_6) \longrightarrow w_6$$

Next, also by Pareto principle, (v_2, v_1) corresponds to w_1 or w_2 in S^1. First, assume

$$(v_2, v_1) \longrightarrow w_2 \tag{1.1}$$

(1.1) means that when individual A prefers x_3 to x_2 and individual B prefers x_2 to x_3, then the society prefers x_3 to x_2. By Pareto principle, independence of irrelevant alternatives, and transitivity we have

$$(v_4, v_6) \longrightarrow w_4$$

This means that when individual A prefers x_3 to x_1 and individual B prefers x_1 to x_3, then the society prefers x_3 to x_1. Similarly, we get

$$(v_5, v_1) \longrightarrow w_5$$

This means that when individual A prefers x_2 to x_1 and individual B prefers x_1 to x_2, then the society prefers x_2 to x_1. Similarly, we get

$$(v_6, v_2) \longrightarrow w_6$$

This means that when individual A prefers x_2 to x_3 and individual B prefers x_3 to x_2, then the society prefers x_2 to x_3. Similarly, we get

$$(v_1, v_3) \longrightarrow w_1$$

This means that when individual A prefers x_1 to x_3 and individual B prefers x_3 to x_1, then the society prefers x_1 to x_3. Similarly, we get

$$(v_2, v_4) \longrightarrow w_2$$

This means that when individual A prefers x_1 to x_2 and individual B prefers x_2 to x_1, then the society prefers x_1 to x_2. These correspondences imply that individual A is the dictator. Therefore, if there is no dictator, we must have

$$(v_2, v_1) \longrightarrow w_1$$

This means that when individual A prefers x_3 to x_2 and individual B prefers x_2 to x_3, then the society prefers x_2 to x_3. By Pareto principle and independence of irrelevant alternatives we get

$$(v_3, v_1) \longrightarrow w_1$$

This means that when individual A prefers x_3 to x_1 and individual B prefers x_1 to x_3, then the society prefers x_1 to x_3. Similarly, we get

$$(v_4, v_2) \longrightarrow w_2$$

This means that when individual A prefers x_2 to x_1 and individual B prefers x_1 to x_2, then the society prefers x_1 to x_2. Then, by Pareto principle and independence of irrelevant alternatives we get correspondences from preference profiles to the social preference when the preference of individual B is fixed at v_1 as follows.

$$(v_4, v_1) \longrightarrow w_1,\ (v_5, v_1) \longrightarrow w_1,\ (v_6, v_1) \longrightarrow w_1$$

Therefore, correspondences from the vertices of A to the vertices of S^1 are obtained as follows.

$$(v_1, v_1) \longrightarrow w_1,\ (v_2, v_1) \longrightarrow w_1,\ (v_3, v_1) \longrightarrow w_1$$

$$(v_4, v_1) \longrightarrow w_1,\ (v_5, v_1) \longrightarrow w_1,\ (v_6, v_1) \longrightarrow w_1$$

By similar logic, if individual B is not a dictator, correspondences from the vertices of B to the vertices of S^1 are obtained as follows.

$$(v_1, v_1) \longrightarrow w_1,\ (v_1, v_2) \longrightarrow w_1,\ (v_1, v_3) \longrightarrow w_1$$

$$(v_1, v_4) \longrightarrow w_1,\ (v_1, v_5) \longrightarrow w_1,\ (v_1, v_6) \longrightarrow w_1$$

Sets of simplices which are 1-dimensional cycles of $\Delta \cup A \cup B$ are only the following z and its counterpart $-z$.

$$\begin{aligned} z = &< (v_1, v_1), (v_2, v_1) > + < (v_2, v_1), (v_3, v_1) > + < (v_3, v_1), (v_4, v_1) > \\ &+ < (v_4, v_1), (v_5, v_1) > + < (v_5, v_1), (v_6, v_1) > + < (v_6, v_1), (v_1, v_1) > \\ &+ < (v_1, v_1), (v_1, v_2) > + < (v_1, v_2), (v_1, v_3) > + < (v_1, v_3), (v_1, v_4) > \\ &+ < (v_1, v_4), (v_1, v_5) > + < (v_1, v_5), (v_1, v_6) > + < (v_1, v_6), (v_1, v_1) > \\ &+ < (v_1, v_1), (v_6, v_6) > + < (v_6, v_6), (v_5, v_5) > + < (v_5, v_5), (v_4, v_4) > \\ &+ < (v_4, v_4), (v_3, v_3) > + < (v_3, v_3), (v_2, v_2) > + < (v_2, v_2), (v_1, v_1) > \end{aligned}$$

Since $\Delta \cup A \cup B$ has no 2-dimensional simplex, z is a representative element of homology classes of $\Delta \cup A \cup B$. z is transferred by the homomorphism of homology groups F_* induced by F to the following

z' in S^1.

$$
\begin{aligned}
z' = &< w_1, w_1 > + < w_1, w_1 > + < w_1, w_1 > + < w_1, w_1 > + < w_1, w_1 > \\
&+ < w_1, w_1 > + < w_1, w_1 > + < w_1, w_1 > + < w_1, w_1 > + < w_1, w_1 > \\
&+ < w_1, w_1 > + < w_1, w_1 > + < w_1, w_6 > + < w_6, w_5 > + < w_5, w_4 > \\
&+ < w_4, w_3 > + < w_3, w_2 > + < w_2, w_1 > \\
= &< w_1, w_6 > + < w_6, w_5 > + < w_5, w_4 > + < w_4, w_3 > + < w_3, w_2 > \\
&+ < w_2, w_1 >
\end{aligned}
$$

This is a cycle of S^1. Therefore, the homology group induced by $(F_*)|_{\Delta \cup A \cup B}$, which is the homomorphism of homology groups induced by $F|_{\Delta \cup A \cup B}$, is not trivial, and hence the degree of mapping of $F|_{\Delta \cup A \cup B}$ is not zero. □

From Theorem 1.1 we obtain the following result.

Theorem 1.2 The non-existence of social welfare function which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator (the Arrow impossibility theorem) is equivalent to the Brouwer fixed point theorem.

1.5 Concluding remarks

We have shown that with two individuals and three alternatives the Arrow impossibility theorem is equivalent to the Brouwer fixed point theorem on a 2-dimensional ball (circle) using elementary concepts and techniques of algebraic topology, in particular, homology groups of simplicial complexes, homomorphisms of homology groups.

Our approach may be applied to other social choice problems such as Wilson's impossibility theorem (Wilson (1972)), the Gibbard-Satterthwaite theorem (Gibbard (1973) and Satterthwaite (1975)) and Amartya Sen's liberal paradox (Sen (1979)).

Chapter 2

A topological approach to the Arrow impossibility theorem when individual preferences are weak orders

We will present a topological approach to the Arrow impossibility theorem of social choice theory that there exists no binary social choice rule (which we will call a social welfare function) which satisfies the conditions of transitivity, independence of irrelevant alternatives (IIA), Pareto principle and non-existence of dictator. Our research is in line with the studies of topological approaches to discrete social choice problems initiated by Baryshnikov (1993). But tools and techniques of algebraic topology which we will use are more elementary than those in Baryshnikov (1993). Our main tools are homology groups of simplicial complexes. And we will consider the case where individual preferences are weak orders, that is, individuals may be indifferent about any pair of alternatives. This point is an extension of the analysis by Baryshnikov (1993)*1.

2.1 Introduction

Topological approaches to social choice problems have been initiated by Chichilnisky (1980). In her model a space of alternatives is a subset of a Euclidean space, and individual preferences over this set are represented by normalized gradient fields. Her main result is an impossibility theorem that there exists no *continuous* social choice rule which satisfies *unanimity* and *anonymity*. This approach has been further developed by Chichilnisky (1979), (1982), Koshevoy (1997), Lauwers (2004), Weinberger (2004), and so on. On the other hand, Baryshnikov (1993) and (1997) have presented a topological approach to Arrow's general possibility theorem, which is usually called the *Arrow impossibility theorem* (Arrow (1963)), in a discrete framework of social choice*2. But he used an advanced concept of algebraic topology, *nerve of a covering*. It is not dealt with in most elementary textbooks of algebraic topology, and is difficult of access for most economists. And he considered only the case where individual preferences are strict, that is, individuals are never indifferent about any pair of alternatives. In this chapter we will attempt a more simple and elementary topological approach to the Arrow impossibility theorem under the assumption

*1 This chapter is based on my paper of the same title published in *Applied Mathematics and Computation*, Vol. 174, No. 2, pp. 961-981, 2006, Elsevier.

*2 About surveys and basic results of topological social choice theories, see Mehta (1997) and Lauwers (2000).

of the *free triple property*. Our main tools are homology groups of simplicial complexes. It is a basic concept of algebraic topology, and is dealt with in almost all elementary textbooks in this field. And we will consider the case where individual preferences are weak orders, that is, individuals may be indifferent about any pair of alternatives. This point is an extension of the analysis by Baryshnikov (1993).

Mainly we will show the following results.

1. Let Δ be an inclusion map from the set of individual preferences to the set of the social preference. Let i_i be an inclusion map from the set of the preference of individual i (a representative individual) to the set of the social preference, and F be a transitive binary social choice rule (which we will call a social welfare function). Let $(F \circ \Delta)_*$ and $(F \circ i_i)_*$ be homomorphisms of homology groups induced by the composite functions of these inclusion maps and F[*3]. Then, we will obtain the following results.

$$(F \circ \Delta)_* = \sum_{i=1}^{k}(F \circ i_i)_* \ (k \text{ is the number of individuals})$$

$$(F \circ \Delta)_* \neq 0$$

2. On the other hand, if social welfare functions satisfy the conditions of Pareto principle, independence of irrelevant alternatives (IIA) as well as transitivity and non-existence of dictator, we can show

$$(F \circ i_i)_* = 0 \text{ for all } i$$

(1) and (2) contradict. Therefore, there exists no binary social choice rule which satisfies transitivity, Pareto principle, IIA and non-existence of dictator.

In the next section we present our model and calculate the homology groups of simplicial complexes which represent individual preferences. In Section 2.3 we will prove the main results.

2.2 The model and simplicial complexes

There are $n(\geq 3)$ alternatives and $k(\geq 2)$ individuals. n and k are finite positive integers. Denote individual i's preference by p_i. A combination of individual preferences, which is called a preference profile, is denoted by $\mathbf{p}$, and the set of preference profiles is denoted by $\mathcal{P}^k$. The alternatives are represented by x_i, $i = 1, 2, \cdots, n$. Individual preferences over the alternatives are weak orders, that is, individuals strictly prefer one alternative to another, or are indifferent between them. We consider a social choice rule which determines a social preference corresponding to a preference profile. Transitive social choice rule is called a *social welfare function* and is denoted by $F(\mathbf{p})$. We assume the free triple property, that is, for each combination of three alternatives individual preferences are never restricted.

Social welfare functions must satisfy Pareto principle and independence of irrelevant alternatives (IIA) as well as transitivity. The meanings of the latter two conditions are as follows.

Pareto principle When all individuals prefer an alternative x_i to another alternative x_j, the society must prefer x_i to x_j.

[*3] A homomorphism h is a mapping from a group A to another group B which satisfies $h(x + y) = h(x) + h(y)$ for $x \in A$, $y \in B$.

Independence of irrelevant alternatives (IIA) The social preference about every pair of two alternatives x_i and x_j is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x_i and x_j.

The Arrow impossibility theorem states that there exists no binary social choice rule which satisfies the conditions of transitivity, IIA, Pareto principle and non-existence of dictator. Dictator is an individual whose strict preference always coincides with the social preference.

Hereafter we will consider a set of alternatives x_1, x_2 and x_3. From the set of individual preferences about x_1, x_2 and x_3 we construct a simplicial complex by the following procedures.

1. A preference of an individual such that he prefers x_1 to x_2 is denoted by $(1,2)$, a preference such that he prefers x_2 to x_1 by $(2,1)$, a preference such that he is indifferent between x_1 and x_2 by $\overline{(1,2)}$, and similarly for other pairs of alternatives. Define vertices of the simplicial complex corresponding to (i,j) and $\overline{(i,j)}$.
2. A line segment between the vertices (i,j) and (k,l) is included in the simplicial complex if and only if the preference represented by (i,j) and the preference represented by (k,l) are consistent, that is, they satisfy transitivity. For example, the line segment between $(1,2)$ and $(2,3)$ is included, but the line segment between $(1,2)$ and $(2,1)$ is not included in the simplicial complex. The line segment between $\overline{(1,2)}$ and $(2,3)$ is included, but the line segment between $(1,2)$ and $\overline{(1,2)}$ is not included in the simplicial complex.
3. A triangle (circumference plus interior) made by three vertices (i,j), (k,l) and (m,n) is included in the simplicial complex if and only if the preferences represented by (i,j), (k,l) and (m,n) satisfy transitivity. For example, since the preferences represented by $(1,2)$, $(2,3)$ and $(1,3)$ satisfy transitivity, a triangle made by these three vertices is included in the simplicial complex. But, since the preferences represented by $(1,2)$, $(2,3)$ and $(3,1)$ do not satisfy transitivity, a triangle made by these three vertices is not included in the simplicial complex. Similar for triangles which include a vertex $\overline{(i,j)}$. Since the preferences represented by $\overline{(1,2)}$, $(2,3)$ and $(1,3)$ satisfy transitivity, a triangle made by these three vertices is included in the simplicial complex. But, since the preferences represented by $\overline{(1,2)}$, $(2,3)$ and $(3,1)$ do not satisfy transitivity, a triangle made by these three vertices is not included in the simplicial complex.

The simplicial complex constructed by these procedures is denoted by P.

In Figure 1 the simplicial complex made by preferences which do not include indifference is depicted. This is called C_1. It is *homotopic* to a circumference of a circle (a 1-dimensional sphere S^1). The simplicial complex made by preferences which may include indifference is constructed by adding the following simplicial complexes to C_1.

The triangle made by $\overline{(1,2)}$, $(2,3)$, $(1,3)$ and its edges and vertices

The triangle made by $\overline{(1,2)}$, $(3,2)$, $(3,1)$ and its edges and vertices

The triangle made by $\overline{(1,3)}$, $(1,2)$, $(3,2)$ and its edges and vertices

The triangle made by $\overline{(1,3)}$, $(2,1)$, $(2,3)$ and its edges and vertices

The triangle made by $\overline{(2,3)}$, $(1,2)$, $(1,3)$ and its edges and vertices

The triangle made by $\overline{(2,3)}$, $(2,1)$, $(3,1)$ and its edges and vertices

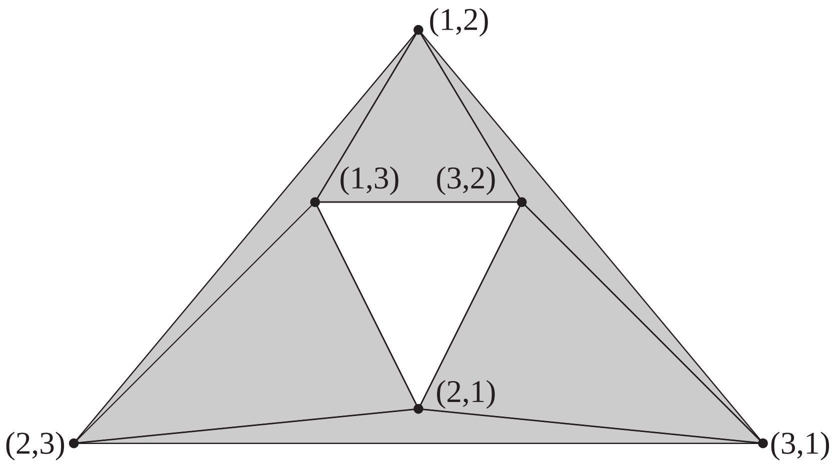

Figure 1: The simplicial complex made by preferences not including indifference (C_1)

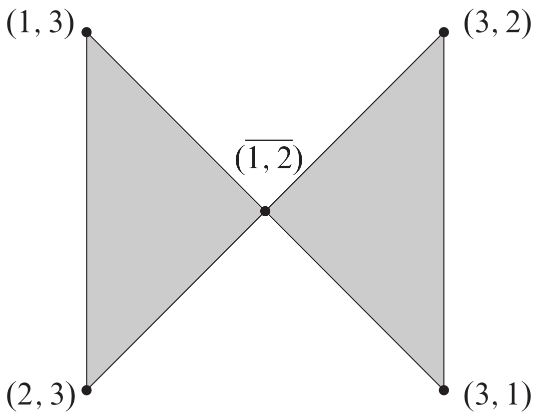

Figure 2: C_2

The triangle made by $\overline{(1,2)}, \overline{(2,3)}, \overline{(1,3)}$ and its edges and vertices

The first two simplicial complexes are depicted in Figure 2. This is called C_2. The latter five simplicial complexes are depicted in Figure 3. This is called D. Let us denote $C = C_1 \cup C_2$.

P is the union of C and D. The intersection of C and D is the graph depicted in Figure 4. This is homotopic to isolated three points. It is denoted by E. Its 0-dimensional homology group is isomorphic to the group of three integers, and its 1-dimensional homology group is trivial, that is, $H_0(E) = \mathbb{Z}^3$ and $H_1(E) = 0$.

Now, we can show the following lemma.

Lemma 2.1 The 1-dimensional homology group of P is isomorphic to the group of 6 integers, that is, $H_1(P) \cong \mathbb{Z}^6$.

Proof. P contains the following 1-dimensional simplices.

$$\sigma_1 =< (1,2),(2,3) >,\ \sigma_2 =< (1,2),(3,2) >,\ \sigma_3 =< (1,2),(1,3) >$$

$$\sigma_4 =< (1,2),(3,1) >,\ \sigma_5 =< (2,1),(2,3) >,\ \sigma_6 =< (2,1),(3,2) >$$

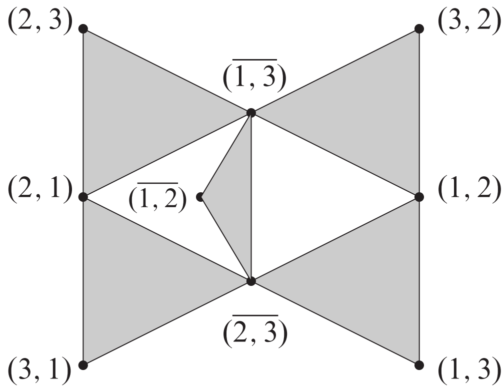

Figure 3: D

$$\sigma_7 =< (2,1),(1,3) >,\ \sigma_8 =< (2,1),(3,1) >,\ \sigma_9 =< (2,3),(1,3) >$$

$$\sigma_{10} =< (2,3),(3,1) >,\ \sigma_{11} =< (3,2),(1,3) >,\ \sigma_{12} =< (3,2),(3,1) >$$

$$\sigma_{13} =< (\overline{1,2}),(2,3) >,\ \sigma_{14} =< (\overline{1,2}),(3,2) >,\ \sigma_{15} =< (\overline{1,2}),(1,3) >$$

$$\sigma_{16} =< (\overline{1,2}),(3,1) >,\ \sigma_{17} =< (\overline{2,3}),(1,2) >,\ \sigma_{18} =< (\overline{2,3}),(2,1) >$$

$$\sigma_{19} =< (\overline{2,3}),(1,3) >,\ \sigma_{20} =< (\overline{2,3}),(3,1) >,\ \sigma_{21} =< (\overline{1,3}),(1,2) >$$

$$\sigma_{22} =< (\overline{1,3}),(2,1) >,\ \sigma_{23} =< (\overline{1,3}),(2,3) >,\ \sigma_{24} =< (\overline{1,3}),(3,2) >$$

$$\sigma_{25} =< (\overline{1,2}),(\overline{2,3}) >,\ \sigma_{26} =< (\overline{1,2}),(\overline{1,3}) >,\ \sigma_{27} =< (\overline{2,3}),(\overline{1,3}) >$$

An element of the 1-dimensional chain group of P is written as follows.

$$c_1(P) = \sum_{i=1}^{27} a_i \sigma_i \tag{2.1}$$

$a_1, a_2, \cdots, a_{27}$ are integers.

From this we obtain

$$\begin{aligned}
\partial c_1(P) =&(-a_1 - a_2 - a_3 - a_4 + a_{17} + a_{21}) < (1,2) > \\
&+ (-a_5 - a_6 - a_7 - a_8 + a_{18} + a_{22}) < (2,1) > \\
&+ (a_1 + a_5 - a_9 - a_{10} + a_{13} + a_{23}) < (2,3) > \\
&+ (a_2 + a_6 - a_{11} - a_{12} + a_{14} + a_{24}) < (3,2) > \\
&+ (a_3 + a_7 + a_9 + a_{11} + a_{15} + a_{19}) < (1,3) > \\
&+ (a_4 + a_8 + a_{10} + a_{12} + a_{16} + a_{20}) < (3,1) > \\
&+ (-a_{13} - a_{14} - a_{15} - a_{16} - a_{25} - a_{26}) < (\overline{1,2}) > \\
&+ (-a_{17} - a_{18} - a_{19} - a_{20} + a_{25} - a_{27}) < (\overline{2,3}) > \\
&+ (-a_{21} - a_{22} - a_{23} - a_{24} + a_{26} + a_{27}) < (\overline{1,3}) >
\end{aligned}$$

The conditions for an element of the 1-dimensional chain group of P, $c_1(P)$, to be a cycle is $\partial c_1(P) = 0$. For this condition to hold all coefficients of $\partial c_1(P)$ must be zero, and we obtain the following equations.

$$-a_1 - a_2 - a_3 - a_4 + a_{17} + a_{21} = 0,\ -a_5 - a_6 - a_7 - a_8 + a_{18} + a_{22} = 0$$

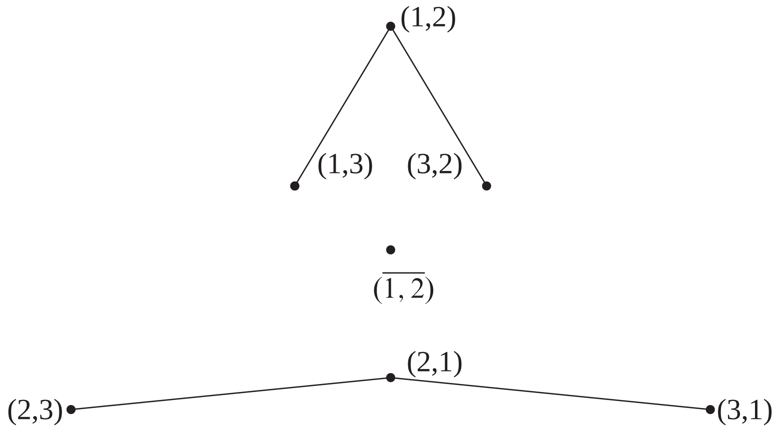

Figure 4: E: (The intersection of C and D)

$$a_1 + a_5 - a_9 - a_{10} + a_{13} + a_{23} = 0,\ a_2 + a_6 - a_{11} - a_{12} + a_{14} + a_{24} = 0$$

$$a_3 + a_7 + a_9 + a_{11} + a_{15} + a_{19} = 0,\ a_4 + a_8 + a_{10} + a_{12} + a_{16} + a_{20} = 0$$

$$-a_{13} - a_{14} - a_{15} - a_{16} - a_{25} - a_{26} = 0,\ -a_{17} - a_{18} - a_{19} - a_{20} + a_{25} - a_{27} = 0$$

$$-a_{21} - a_{22} - a_{23} - a_{24} + a_{26} + a_{27} = 0$$

Summing up the first 8 equations side by side we get the last equation. Therefore, only 8 equations are independent, and we can freely choose the values of 19 variables among $a_1, a_2, \cdots, a_{27}$. Thus, the 1-dimensional cycle group of P, $Z_1(P)$, is isomorphic to the group of 19 integers, that is, $Z_1(P) \cong \mathbb{Z}^{19}$.

P contains the following 2-dimensional simplices.

$$\tau_1 =< (1,2), (2,3), (1,3) >,\ \tau_2 =< (1,2), (3,2), (3,1) >$$

$$\tau_3 =< (1,2), (3,2), (1,3) >,\ \tau_4 =< (2,1), (2,3), (1,3) >$$

$$\tau_5 =< (2,1), (3,2), (3,1) >,\ \tau_6 =< (2,1), (2,3), (3,1) >$$

$$\tau_7 =< (\overline{1,2}), (2,3), (1,3) >,\ \tau_8 =< (\overline{1,2}), (3,2), (3,1) >$$

$$\tau_9 =< (\overline{2,3}), (1,2), (1,3) >,\ \tau_{10} =< (\overline{2,3}), (2,1), (3,1) >$$

$$\tau_{11} =< (\overline{1,3}), (1,2), (3,2) >,\ \tau_{12} =< (\overline{1,3}), (2,1), (2,3) >$$

$$\tau_{13} =< (\overline{1,2}), (\overline{2,3}), (\overline{1,3}) >$$

An element of the 2-dimensional chain group of P is written as follows.

$$c_2(P) = \sum_{i=1}^{13} b_i \tau_i$$

$b_1, b_2, \cdots, b_{13}$ are integers. The image of the boundary homomorphism of the 2-dimensional chain group of P is

$$\begin{aligned}\partial c_2(P) =& \sum_{i=1}^{13} b_i \partial \tau_i \\ =& b_1\sigma_1 + (b_2 + b_3 + b_{11})\sigma_2 + (-b_1 - b_3 + b_9)\sigma_3 - b_2\sigma_4 + (b_4 + b_6 + b_{12})\sigma_5 \\ &+ b_5\sigma_6 - b_4\sigma_7 - (b_5 + b_6)\sigma_8 + (b_1 + b_4 + b_7)\sigma_9 + b_6\sigma_{10} \\ &+ b_3\sigma_{11} + (b_2 + b_5 + b_8)\sigma_{12} + b_7\sigma_{13} + b_8\sigma_{14} - b_7\sigma_{15} \\ &- b_8\sigma_{16} + b_9\sigma_{17} + b_{10}\sigma_{18} - b_9\sigma_{19} - b_{10}\sigma_{20} + b_{11}\sigma_{21} \\ &+ b_{12}\sigma_{22} - b_{12}\sigma_{23} - b_{11}\sigma_{24} + b_{13}\sigma_{25} - b_{13}\sigma_{26} - b_{13}\sigma_{27} \end{aligned} \tag{2.2}$$

The values of the coefficients of $\sigma_1, \sigma_2, \sigma_3, \sigma_5, \sigma_6, \sigma_8, \sigma_9, \sigma_{12}, \sigma_{17}, \sigma_{18}, \sigma_{21}, \sigma_{22}, \sigma_{25}$ are determined by $b_1, b_2, \cdots, b_{13}$, and then the values of other σ's are also determined. Thus, the 1-dimensional boundary cycle group of P, $B_1(P)$, is isomorphic to the group of 13 integers, that is, $B_1(P) \cong \mathbb{Z}^{13}$. Therefore, the 1-dimensional homology group of P is isomorphic to the group of 6 integers, that is, we obtain $H_1(P) = Z_1(P)/B_1(P) \cong \mathbb{Z}^6$. □

Next we consider the simplicial complex, P^k, made by the set of preference profiles of individuals, $\mathcal{P}^k$, about x_1, x_2 and x_3. We can show the following result.

Lemma 2.2 The 1-dimensional homology group of P^k is isomorphic to the group of $6k$ integers, that is, $H_1(P^k) \cong \mathbb{Z}^{6k}$.

Proof. As a preliminary result, we show $H_1(P \times C) \cong \mathbb{Z}^8$. Using C_1^1, C_1^2, C_2^1 and C_2^2 depicted in Figure 5 and 6[*4], C is represented as $C = C^1 \cup C^2$, $C^1 = C_1^1 \cup C_2^1$, $C^2 = C_1^2 \cup C_2^2$. C^1 and C^2 are homotopic to one point, and the intersection of C^1 and C^2 consists of two segments and one point, which is denoted by G. G is homotopic to three isolated points, and we have $H_1(G) = 0$ and $H_0(G) \cong \mathbb{Z}^3$. From these arguments we obtain the following Mayer-Vietoris exact sequence[*5]

$$H_1(P \times G)(\cong (\mathbb{Z}^6)^3) \xrightarrow{k_1} H_1(P \times C^1) \oplus H_1(P \times C^2)(\cong \mathbb{Z}^6 \oplus \mathbb{Z}^6) \xrightarrow{w_1} H_1(P \times C) \longrightarrow$$
$$\xrightarrow{\alpha_1} H_0(P \times G)(\cong \mathbb{Z}^3) \xrightarrow{k_0} H_0(P \times C^1) \oplus H_0(P \times C^2)(\cong \mathbb{Z} \oplus \mathbb{Z}) \longrightarrow$$
$$\xrightarrow{w_0} H_0(P \times C)(\cong \mathbb{Z}) \longrightarrow 0$$

Since w_0 is a surjection (onto mapping)[*6], we have Image $w_0 \cong \mathbb{Z}$. By the homomorphism theorem we obtain $H_0(P \times C^1) \oplus H_0(P \times C^2)/\text{Ker } w_0 \cong \mathbb{Z}$, and then Ker $w_0 \cong \mathbb{Z}$ is derived. Thus, from the condition of exact sequences we have Image $k_0 \cong$ Ker $w_0 \cong \mathbb{Z}$. Again by the homomorphism theorem we obtain $H_0(P \times G)/\text{Ker } k_0 \cong$ Image $k_0 \cong \mathbb{Z}$, and we get Ker $k_0 \cong \mathbb{Z} \oplus \mathbb{Z}$. Thus, we have Image $\alpha_1 \cong$ Ker $k_0 \cong \mathbb{Z} \oplus \mathbb{Z}$, and by the homomorphism theorem $H_1(P \times C)/\text{Ker } \alpha_1 \cong \mathbb{Z} \oplus \mathbb{Z}$ is derived. From the condition of exact sequences we have Ker $\alpha_1 \cong$ Image w_1, and by the homomorphism theorem, $H_1(P \times C^1) \oplus H_1(P \times C^2)/\text{Ker } w_1 \cong$ Image w_1 is derived. From the condition of exact sequences we obtain Ker $w_1 \cong$ Image k_1. Now let us consider Image k_1.

[*4] C_1^1 and C_1^2 are depicted in Figure 5, and C_2^1 and C_2^2 are depicted in Figure 6.

[*5] About homology groups, the homomorphism theorem and the Mayer-Vietoris exact sequences we referred to Tamura (1970) and Komiya (2001).

[*6] This is derived from the condition of exact sequences.

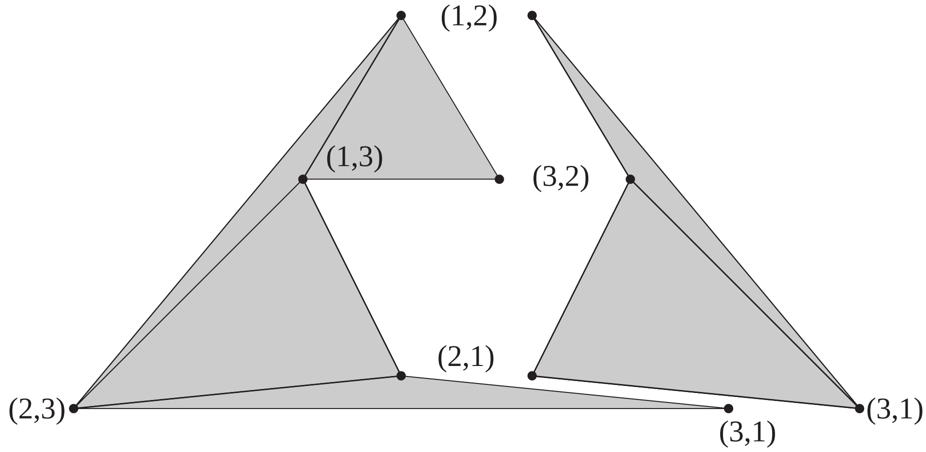

Figure 5: C_1^1 and C_1^2

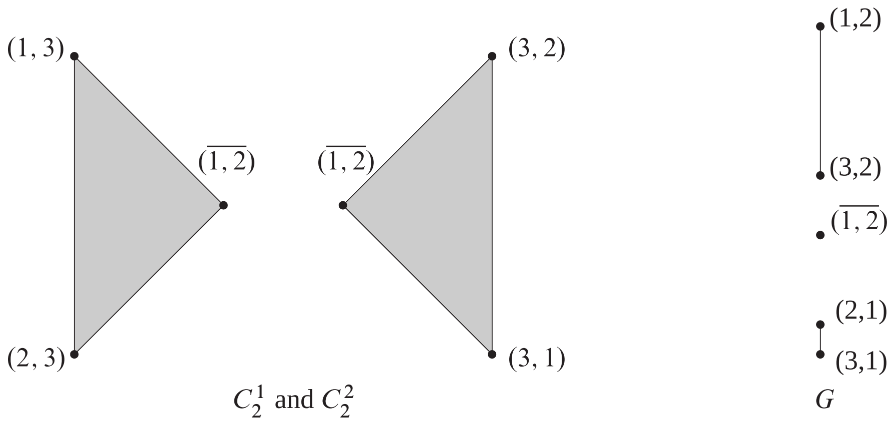

Figure 6: C_2^1, C_2^2 and G

Let x, y, z be the vertices of three connected components of G. Let $h \in H_1(P)$, then $h \times x \in H_1(P \times x)$, $h \times y \in H_1(P \times y)$ and $h \times z \in H_1(P \times z)$ belong to the different homology classes. Since C^1 is connected, there exists a sequence of 1-dimensional simplices connected x and y, and a sequence of 1-dimensional simplices connected x and z. Thus, they belong to the same homology class in $H_1(P \times C^1)$. We can show a similar result for $H_1(P \times C^2)$. Therefore we obtain Image $k_1 \cong \mathbb{Z}^6$.

From Ker $w_1 \cong$ Image k_1 we have Ker $w_1 \cong \mathbb{Z}^6$, and from $H_1(P \times C^1) \oplus H_1(P \times C^2)/\mathrm{Ker}\, w_1 \cong$ Image w_1 we have Image $w_1 \cong \mathbb{Z}^6$. Thus, Ker $\alpha_1 \cong \mathbb{Z}^6$ is derived. Therefore, we obtain $H_1(P \times C) \cong \mathbb{Z}^8$. By similar procedures we can show $H_1(P \times D) \cong \mathbb{Z}^8$.

Using this result we will show $H_1(P^2) \cong \mathbb{Z}^{12}$. Since $P^2 = P \times (C \cup D) = (P \times C) \cup (P \times D)$, and $(P \times C) \cap (P \times D) = P \times E$ we obtain the following Mayer-Vietoris exact sequence.

$$H_1(P \times E)(\cong (\mathbb{Z}^6)^3) \xrightarrow{k_1} H_1(P \times C) \oplus H_1(P \times D)(\cong \mathbb{Z}^8 \oplus \mathbb{Z}^8) \xrightarrow{w_1} H_1(P^2) \longrightarrow$$

$$\xrightarrow{\alpha_1} H_0(P \times E)(\cong \mathbb{Z}^3) \xrightarrow{k_0} H_0(P \times C) \oplus H_0(P \times D)(\cong \mathbb{Z} \oplus \mathbb{Z}) \longrightarrow$$

$$\xrightarrow{w_0} H_0(P^2)(\cong \mathbb{Z}) \longrightarrow 0$$

Since w_0 is a surjection, we have Image $w_0 \cong \mathbb{Z}$. By the homomorphism theorem we obtain $H_0(P\times C)\oplus H_0(P\times D)/\text{Ker } w_0 \cong \mathbb{Z}$, and Ker $w_0 \cong \mathbb{Z}$ is derived. Thus, from the condition of exact sequences we have Image $k_0 \cong$ Ker $w_0 \cong \mathbb{Z}$. Again by the homomorphism theorem we obtain $H_0(P\times E)/\text{Ker } k_0 \cong$ Image $k_0 \cong \mathbb{Z}$, and Ker $k_0 \cong \mathbb{Z}\oplus\mathbb{Z}$ is derived. Thus, we have Image $\alpha_1 \cong$ Ker $k_0 \cong \mathbb{Z}\oplus\mathbb{Z}$, and by the homomorphism theorem $H_1(P^2)/\text{Ker } \alpha_1 \cong \mathbb{Z}\oplus\mathbb{Z}$ is derived. Again from the condition of exact sequences we obtain Ker $\alpha_1 \cong$ Image w_1, and by the homomorphism theorem we obtain $H_1(P\times C)\oplus H_1(P\times D)/\text{Ker } w_1 \cong$ Image w_1. Further, from the condition of exact sequences Ker $w_1 \cong$ Image k_1 is derived. Now consider Image k_1.

Let x, y, z be the vertices of the connected components of E. Let $h\in H_1(P)$, then $h\times x\in H_1(P\times x)$, $h\times y\in H_1(P\times y)$ and $h\times z\in H_1(P\times z)$ belong to different homology classes. But, since C is connected, there exists a sequence of 1-dimensional simplices connecting x and y, and a sequence of 1-dimensional simplices connecting x and z. Thus, they belong to the same homology class in $H_1(P\times C)$. Similar for $H_1(P\times D)$. Therefore, we obtain Image $k_1 \cong \mathbb{Z}^6$.

From Ker $w_1 \cong$ Image k_1 we have Ker $w_1 \cong \mathbb{Z}^6$. And from $H_1(P\times C)\oplus H_1(P\times D)/\text{Ker } w_1 \cong$ Image w_1 we obtain Image $w_1 \cong \mathbb{Z}^{10}$. Thus, Ker $\alpha_1 \cong \mathbb{Z}^{10}$ is derived. Therefore, we get $H_1(P^2)\cong\mathbb{Z}^{12}$.

Inductively we can show $H_1(P^k)\cong\mathbb{Z}^{6k}$. □

The social preference is also represented by P. The social preference about x_i and x_j is (i,j) or (j,i) or $\overline{(i,j)}$. By the condition of IIA, individual preferences about alternatives other than x_i and x_j do not affect the social preference about them. Thus, the social welfare function F is a function from the vertices in P^k to the vertices in P. A set of points in P^k spans a simplex if and only if individual preferences represented by these points are consistent, that is, they satisfy transitivity, and the social preference derived from the profile represented by these points also satisfies transitivity. Therefore, if a set of points in P^k spans a simplex, the set of points in P which represent the social preference corresponding to these points in P^k also spans a simplex in P, and hence the social welfare function is a *simplicial map*. It is naturally extended from the vertices in P^k to all points in P^k. Each point in P^k is represented as a convex combination of the vertices in P^k. This function is also denoted by F. When P represents the social preference, we denote it by P_s. Then, F is defined as a function from P^k to P_s.

We define an inclusion map from P to P^k, $\Delta: P\longrightarrow P^k: p\longrightarrow(p,p,\cdots,p)$, and an inclusion map which is derived by fixing preferences of individuals other than individual l to $\mathbf{p}_{-l}$, $i_l: P\longrightarrow P^k: p\longrightarrow(\mathbf{p}_{-l},p)$. The homomorphisms of 1-dimensional homology groups induced by these inclusion maps are

$$\Delta_*: \mathbb{Z}^6\longrightarrow\mathbb{Z}^{6k}: h\longrightarrow(h,h,\cdots,h),\ h\in\mathbb{Z}^6$$

$$i_{l*}: \mathbb{Z}^6\longrightarrow\mathbb{Z}^{6k}: h\longrightarrow(0,\cdots,h,\cdots,0)\ \text{(only the } l\text{-th component is } h \text{ and others are zero, } h\in\mathbb{Z}^6)$$

From these definitions about Δ_* and i_{l*} we obtain the following relation.

$$\Delta_* = i_{1*}+i_{2*}+\cdots+i_{n*} \tag{2.3}$$

And the homomorphism of homology groups induced by F is represented as follows.

$$F_*: \mathbb{Z}^{6k}\longrightarrow\mathbb{Z}^6: \mathbf{h}=(h_1,h_2,\cdots,h_n)\longrightarrow h,\ h\in\mathbb{Z}^6$$

The composite function of i_l and the social welfare function F is $F \circ i_l : P \longrightarrow P_s$, and its induced homomorphism satisfies $(F \circ i_l)_* = F_* \circ i_{l*}$. The composite function of Δ and F is $F \circ \Delta : P \longrightarrow P_s$, and its induced homomorphism satisfies $(F \circ \Delta)_* = F_* \circ \Delta_*$. From (2.3) we have

$$(F \circ \Delta)_* = (F \circ i_1)_* + (F \circ i_2)_* + \cdots (F \circ i_n)_*$$

$F \circ i_l$ when a preference profile of individuals other than individual l is $\mathbf{p}_{-l}$ and $F \circ i_l$ when a preference profile of individuals other than individual l is $\mathbf{p}'_{-l}$ are homotopic. Thus, the induced homomorphism $(F \circ i_l)_*$ of $F \circ i_l$ does not depend on the preferences of individuals other than l.

Note: Let $F \circ i_l(\mathbf{p}_{-l}, p_l)$ be the composite function of i_l and F when the preference profile of individuals other than l is $\mathbf{p}_{-l}$, and $F \circ i_l(\mathbf{p}'_{-l}, p_l)$ be the composite function of i_l and F when the preference profile of individuals other than l is $\mathbf{p}'_{-l}$. The component for one individual (denoted by k) of $\mathbf{p}_{-l}$ and that of $\mathbf{p}'_{-l}$ are denoted by p_k and p'_k. His preferences for the pair of alternatives x_i and x_j are denoted by $p_k(i,j)$ and $p'_k(i,j)$. Each of them corresponds to a point (i,j) or (j,i) or $(\overline{i,j})$ in P. Let (m,n) be a point in P such that $p_k(i,j)$ and $p'_k(i,j)$ are different from (m,n), (n,m) and $(\overline{m,n})$. Then, there exists a 1-dimensional simplex (a line segment) between $p_k(i,j)$ and (m,n), and a 1-dimensional simplex between $p'_k(i,j)$ and (m,n). Let

$$p''_k(i,j) = (1-2t)p_k(i,j) + 2t(m,n), \text{ if } 0 \le t < \frac{1}{2}$$
$$p''_k(i,j) = (2t-1)p'_k(i,j) + (2-2t)(m,n), \text{ if } \frac{1}{2} \le t \le 1$$

Then, $p''_k(i,j)$ is a point in P. Let us consider such $p''_k(i,j)$'s for all pairs of alternatives (x_i, x_j), and we denote a set of all $p''_k(i,j)$'s by p''_k. Similarly, p''_k's for all individuals other than k are defined. Let $\mathbf{p}''_{-l}$ be a combination of p''_k's for all individuals other than l, and define

$$H(p,t) = F(\mathbf{p}''_{-l}, p_l)$$

Then, this is a homotopy between $F \circ i_l(\mathbf{p}_{-l}, p_l)$ and $F \circ i_l(\mathbf{p}'_{-l}, p_l)$.

Let $z = <(1,2),(2,3)> + <(2,3),(3,1)> - <(1,2),(3,1)>$ be a cycle of P. By Pareto principle z corresponds to the same cycle in P_s by $(F \circ \Delta)_*$. Since it is not a boundary cycle, we have $(F \circ \Delta)_* \neq 0$.

Note: z is obtained by substituting $a_1 = 1$, $a_4 = -1$, $a_{10} = 1$ and 0 into all other coefficients of an element of the chain group of P expressed in (2.1). For this z to be a boundary of some 2-dimensional simplex we must have $b_1 = b_2 = b_6 = 1$ and $b_i = 0$ for all other coefficients of $\partial c_2(P)$ in (2.2). But then, b_5, b_4, b_3, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}, b_{13} must be 0, and the coefficient of σ_2 is 1. Thus, z is not a boundary cycle.

For a pair of alternatives x_i and x_j, a preference profile, at which all individuals prefer x_i to x_j, is denoted by $(i,j)^{(+,+,\cdots,+)}$; a preference profile, at which they prefer x_j to x_i, is denoted by $(i,j)^{(-,-,\cdots,-)}$. Similarly a preference profile, at which all individuals other than l prefer x_i to x_j, is denoted by $(i,j)_{-l}^{(+,+,\cdots,+)}$; a preference profile, at which they prefer x_j to x_i, is denoted by $(i,j)_{-l}^{(-,-,\cdots,-)}$; a preference profile, at which they are indifferent between x_i and x_j, is denoted by $(i,j)_{-l}^{(0,0,\cdots,0)}$. And a preference profile, at which the preferences of individuals other than l about x_i and x_j are not specified, is denoted by $(i,j)_{-l}^{(?,?,\cdots,?)}$.

2.3 The main results

From preliminary analyses in the previous section we will show the following lemma.

Lemma 2.3 1. If individual l is the dictator, we have

$$(F \circ i_l)_* \cong (F \circ \Delta)_*$$

that is, $(F \circ i_l)_*$ and $(F \circ \Delta)_*$ are isomorphic.

2. If individual l is not a dictator, we have

$$(F \circ i_l)_* = 0$$

Proof. 1. Consider three alternatives x_1, x_2 and x_3 and a preference profile **p** over these alternatives such that the preferences of individuals other than l are represented by $(1,2)_{-l}^{(0,0,\cdots,0)}$, $(2,3)_{-l}^{(0,0,\cdots,0)}$ and $(1,3)_{-l}^{(0,0,\cdots,0)}$, that is, they are indifferent about x_1, x_2 and x_3. If individual l is the dictator, correspondences from his preference to the social preference by $F \circ i_l$ are as follows,

$$(1,2)_l \longrightarrow (1,2),\quad (2,1)_l \longrightarrow (2,1)$$

$$(2,3)_l \longrightarrow (2,3),\quad (3,2)_l \longrightarrow (3,2)$$

$$(1,3)_l \longrightarrow (1,3),\quad (3,1)_l \longrightarrow (3,1)$$

$(1,2)_l$ and $(2,1)_l$ denote the preference of individual l about x_1 and x_2. $(2,3)_l$, $(3,2)_l$ and so on are similar. These correspondences are completely identical to the correspondences by $F \circ \Delta$. Further, since we assume that individuals other than l are indifferent about x_1, x_2 and x_3, correspondences from the preferences of individual l, $\overline{(1,2)}_l$, $\overline{(2,3)}_l$ and $\overline{(1,3)}_l$, to the social preference by $F \circ i_l$ are also identical to the correspondences by $F \circ \Delta$. Therefore, the homomorphism of homology groups, $(F \circ \Delta)_*$ induced by $F \circ \Delta$, and the homomorphism of homology groups, $(F \circ i_l)_*$, which is induced by $F \circ i_l$, are identical (isomorphic), that is, $(F \circ i_l)_* \cong (F \circ \Delta)_*$.

2. Consider three alternatives x_1, x_2 and x_3 and a preference profile **p** over these alternatives such that the preferences of individuals other than l are represented by $(1,2)_{-l}^{(+,+,\cdots,+)}$, $(2,3)_{-l}^{(+,+,\cdots,+)}$ and $(1,3)_{-l}^{(+,+,\cdots,+)}$. If individual l is not a dictator, there exists a preference profile at which the social preference about some pair of alternatives does not coincide with the strict preference of individual l. Assume that when the preference of individual l is $(1,2)$, the social preference is $(2,1)$ or $\overline{(2,1)}$. Then, we obtain the following correspondence from the preference profile to the social preference.

$$(1,2)_{-l}^{(?,?,\cdots,?)} \times (1,2)_l \longrightarrow (2,1) \text{ or } \overline{(2,1)}$$

By Pareto principle we have

$$(1,3)^{(+,+,\cdots,+)} \longrightarrow (1,3)$$

Then, from transitivity we obtain

$$(2,3)_{-l}^{(+,+,\cdots,+)} \times (3,2)_l \longrightarrow (2,3)$$

From Pareto principle we have

$$(1,2)^{(+,+,\cdots,+)} \longrightarrow (1,2)$$

From transitivity we obtain the following correspondence.

$$(1,3)_{-l}^{(+,+,\cdots,+)} \times (3,1)_l \longrightarrow (1,3)$$

Further, from Pareto principle we have

$$(2,3)^{(-,-,\cdots,-)} \longrightarrow (3,2)$$

From transitivity we get the following correspondence.

$$(1,2)_{-l}^{(+,+,\cdots,+)} \times (2,1)_l \longrightarrow (1,2)$$

From these results we find that at the preference profile **p**, where the preferences of individuals other than l are represented by $(1,2)_{-l}^{(+,+,\cdots,+)}$, $(2,3)_{-l}^{(+,+,\cdots,+)}$ and $(1,3)_{-l}^{(+,+,\cdots,+)}$, correspondences from the preference of individual l to the social preference by $F \circ i_l$ are obtained as follows.

$$(1,2)_l \longrightarrow (1,2),\ (2,1)_l \longrightarrow (1,2)$$
$$(2,3)_l \longrightarrow (2,3),\ (3,2)_l \longrightarrow (2,3)$$
$$(1,3)_l \longrightarrow (1,3),\ (3,1)_l \longrightarrow (1,3)$$

From these correspondences with transitivity and IIA we find the following fact.

When individual l is indifferent between x_1 and x_3, the society prefers x_1 to x_3, that is, we obtain the following correspondence.

$$(\overline{1,3})_l \longrightarrow (1,3)$$

This is derived from two correspondences $(1,2)_l \longrightarrow (1,2)$ and $(3,2)_l \longrightarrow (2,3)$. Thus, the following four sets of correspondences are impossible because the correspondences in each set are not consistent with $(\overline{1,3})_l \longrightarrow (1,3)$.

(a) $(\overline{1,2})_l \longrightarrow (\overline{1,2}),\ (\overline{2,3})_l \longrightarrow (\overline{2,3})$
(b) $(\overline{1,2})_l \longrightarrow (\overline{1,2}),\ (\overline{2,3})_l \longrightarrow (3,2)$
(c) $(\overline{1,2})_l \longrightarrow (2,1),\ (\overline{2,3})_l \longrightarrow (3,2)$
(d) $(\overline{1,2})_l \longrightarrow (2,1),\ (\overline{2,3})_l \longrightarrow (\overline{2,3})$

And, we have the following five cases. They are consistent with the correspondence $(\overline{1,3})_l \longrightarrow (1,3)$.

(a) Case (i): $(\overline{1,2})_l \longrightarrow (\overline{1,2}),\ (\overline{2,3})_l \longrightarrow (2,3)$
(b) Case (ii): $(\overline{1,2})_l \longrightarrow (1,2),\ (\overline{2,3})_l \longrightarrow (\overline{2,3})$
(c) Case (iii): $(\overline{1,2})_l \longrightarrow (1,2),\ (\overline{2,3})_l \longrightarrow (2,3)$
(d) Case (iv): $(\overline{1,2})_l \longrightarrow (1,2),\ (\overline{2,3})_l \longrightarrow (3,2)$
(e) Case (v): $(\overline{1,2})_l \longrightarrow (2,1),\ (\overline{2,3})_l \longrightarrow (2,3)$

We consider each case in detail.

(a) Case (i): $(\overline{1,2}) \longrightarrow (\overline{1,2}),\ (\overline{2,3}) \longrightarrow (2,3)$

The vertices mapped by $F \circ i_l$ to the social preference from the preference of individual l span the following five simplices.

$$< (1,2),(2,3) >,\ < (1,2),(1,3) >,\ < (2,3),(1,3) >,\ < (\overline{1,2}),(2,3) >,$$
$$< (\overline{1,2}),(1,3) >$$

Then, an element of the 1-dimensional chain group is written as

$$c_1 = a_1 < (1,2),(2,3) > + a_2 < (1,2),(1,3) > + a_3 < (2,3),(1,3) >$$
$$+ a_4 < (\overline{1,2}),(2,3) > + a_5 < (\overline{1,2}),(1,3) >,\ a_i \in \mathbb{Z}$$

The condition for an element of the 1-dimensional chain group to be a cycle is

$$\partial c_1 = (-a_1 - a_2) < (1,2) > + (a_1 - a_3 + a_4) < (2,3) > + (a_2 + a_3 + a_5) < (1,3) > \\ + (-a_4 - a_5) < (\overline{1,2}) >= 0$$

From this

$$-a_1 - a_2 = 0,\ a_1 - a_3 + a_4 = 0,\ a_2 + a_3 + a_5 = 0,\ -a_4 - a_5 = 0$$

are derived. Then, we obtain $a_2 = -a_1, a_5 = -a_4, a_3 = a_1 + a_4$. Therefore, an element of the 1-dimensional cycle group, Z_1, is written as follows.

$$z_1 = a_1 < (1,2),(2,3) > - a_1 < (1,2),(1,3) > + (a_1 + a_4) < (2,3),(1,3) > \\ + a_4 < (\overline{1,2}),(2,3) > - a_4 < (\overline{1,2}),(1,3) >$$

On the other hand, the vertices span the following 2-dimensional simplices.

$$< (1,2),(2,3),(1,3) >,\ < (\overline{1,2}),(2,3),(1,3) >$$

Then, an element of the 2-dimensional chain group is written as

$$c_2 = b_1 < (1,2),(2,3),(1,3) > + b_2 < (\overline{1,2}),(2,3),(1,3) >,\ b_i \in \mathbb{Z}$$

And an element of the 1-dimensional boundary cycle group, B_1, is written as follows.

$$\partial c_2 = b_1 < (1,2),(2,3) > - b_1 < (1,2),(1,3) > + (b_1 + b_2) < (2,3),(1,3) > \\ + b_2 < (\overline{1,2}),(2,3) > - b_2 < (\overline{1,2}),(1,3) >$$

Then, we find that B_1 is isomorphic to Z_1, and so the 1-dimensional homology group is trivial, that is, we have proved $(F \circ i_l)_* = 0$.

(b) Case (ii): $(\overline{1,2}) \longrightarrow (1,2),\ (\overline{2,3}) \longrightarrow (\overline{2,3})$

The vertices mapped by $F \circ i_l$ to the social preference from the preference of individual l span the following five simplices.

$$< (1,2),(2,3) >,\ < (1,2),(1,3) >,\ < (2,3),(1,3) >,\ < (\overline{2,3}),(1,2) >, \\ < (\overline{2,3}),(1,3) >$$

Then, an element of the 1-dimensional chain group is written as

$$c_1 = a_1 < (1,2),(2,3) > + a_2 < (1,2),(1,3) > + a_3 < (2,3),(1,3) > \\ + a_4 < (\overline{2,3}),(1,2) > + a_5 < (\overline{2,3}),(1,3) >$$

The condition for an element of the 1-dimensional chain group to be a cycle is

$$\partial c_1 = (-a_1 - a_2 + a_4) < (1,2) > + (a_1 - a_3) < (2,3) > + (a_2 + a_3 + a_5) < (1,3) > \\ + (-a_4 - a_5) < (\overline{2,3}) >= 0$$

From this

$$-a_1 - a_2 + a_4 = 0,\ a_1 - a_3 = 0,\ a_2 + a_3 + a_5 = 0,\ -a_4 - a_5 = 0$$

are derived. Then, we obtain $a_3 = a_1, a_5 = -a_4, a_2 = a_4 - a_1$ Therefore, an element of the 1-dimensional cycle group, Z_1, is written as follows.

$$\begin{aligned} z_1 =& a_1 < (1,2),(2,3) > +(a_4 - a_1) < (1,2),(1,3) > +a_1 < (2,3),(1,3) > \\ &+ a_4 < \overline{(1,2)},(2,3) > -a_4 < \overline{(1,2)},(1,3) > \end{aligned}$$

On the other hand, the vertices span the following 2-dimensional simplices.

$$< (1,2),(2,3),(1,3) >, \ < \overline{(2,3)},(1,2),(1,3) >$$

Then, an element of the 2-dimensional chain group is written as

$$c_2 = b_1 < (1,2),(2,3),(1,3) > +b_2 < \overline{(2,3)},(1,2),(1,3) >$$

And an element of the 1-dimensional boundary cycle group, B_1, is written as follows.

$$\begin{aligned} \partial c_2 =& b_1 < (1,2),(2,3) > +(b_2 - b_1) < (1,2),(1,3) > +b_1 < (2,3),(1,3) > \\ &+ b_2 < \overline{(2,3)},(1,2) > -b_2 < \overline{(2,3)},(1,3) > \end{aligned}$$

We find that B_1 is isomorphic to Z_1, and so the 1-dimensional homology group is trivial, that is, we have proved $(F \circ i_l)_* = 0$.

(c) Case (iii): $\overline{(1,2)} \longrightarrow (1,2), \ \overline{(2,3)} \longrightarrow (2,3)$
The vertices mapped by $F \circ i_l$ to the social preference from the preference of individual l span the following three simplices.

$$< (1,2),(2,3) >, \ < (1,2),(1,3) >, \ < (2,3),(1,3) >$$

Then, an element of the 1-dimensional chain group is written as

$$c_1 = a_1 < (1,2),(2,3) > +a_2 < (1,2),(1,3) > +a_3 < (2,3),(1,3) >$$

The condition for an element of the 1-dimensional chain group to be a cycle is

$$\partial c_1 = (-a_1 - a_2) < (1,2) > +(a_1 - a_3) < (2,3) > +(a_2 + a_3) < (1,3) >= 0$$

From this

$$-a_1 - a_2 = 0, \ a_1 - a_3 = 0, \ a_2 + a_3 = 0$$

are derived, and we obtain $a_2 = -a_1, a_3 = a_1$. Therefore, an element of the 1-dimensional cycle group, Z_1, is written as follows.

$$z_1 = a_1 < (1,2),(2,3) > -a_1 < (1,2),(1,3) > +a_1 < (2,3),(1,3) >$$

On the other hand, the vertices span the following 2-dimensional simplex.

$$< (1,2),(2,3),(1,3) >$$

Then, an element of the 2-dimensional chain group is written as

$$c_2 = b_1 < (1,2),(2,3),(1,3) >$$

And an element of the 1-dimensional boundary cycle group, B_1, is written as follows.

$$\partial c_2 = b_1 < (1,2),(2,3) > -b_1 < (1,2),(1,3) > +b_1 < (2,3),(1,3) >$$

We find that B_1 is isomorphic to Z_1, and so the 1-dimensional homology group is trivial, that is, we have proved $(F \circ i_l)_* = 0$.

(d) Case (iv): $\overline{(1,2)} \longrightarrow (1,2),\ \overline{(2,3)} \longrightarrow (3,2)$

The vertices mapped by $F \circ i_l$ to the social preference from the preference of individual l span the following five simplices.

$$< (1,2),(2,3) >,\ < (1,2),(1,3) >,\ < (2,3),(1,3) >,\ < (3,2),(1,2) >,\\ < (3,2),(1,3) >$$

Then, an element of the 1-dimensional chain group is written as

$$\begin{aligned} c_1 =& a_1 < (1,2),(2,3) > + a_2 < (1,2),(1,3) > + a_3 < (2,3),(1,3) > \\ &+ a_4 < (3,2),(1,2) > + a_5 < (3,2),(1,3) > \end{aligned}$$

The condition for an element of the 1-dimensional chain group to be a cycle is

$$\begin{aligned} \partial c_1 =& (-a_1 - a_2 + a_4) < (1,2) > + (a_1 - a_3) < (2,3) > + (a_2 + a_3 + a_5) < (1,3) > \\ &+ (-a_4 - a_5) < (3,2) > = 0 \end{aligned}$$

From this

$$-a_1 - a_2 + a_4 = 0,\ a_1 - a_3 = 0,\ a_2 + a_3 + a_5 = 0,\ -a_4 - a_5 = 0$$

are derived, and we obtain $a_3 = a_1, a_5 = -a_4, a_2 = a_4 - a_1$. Therefore, an element of the 1-dimensional cycle group, Z_1, is written as follows.

$$\begin{aligned} z_1 =& a_1 < (1,2),(2,3) > + (a_4 - a_1) < (1,2),(1,3) > + a_1 < (2,3),(1,3) > \\ &+ a_4 < (3,2),(2,3) > - a_4 < (3,2),(1,3) > \end{aligned}$$

On the other hand, the vertices span the following 2-dimensional simplices.

$$< (1,2),(2,3),(1,3) >,\ < (3,2),(1,2),(1,3) >$$

Then, an element of the 2-dimensional chain group is written as

$$c_2 = b_1 < (1,2),(2,3),(1,3) > + b_2 < (3,2),(1,2),(1,3) >$$

And an element of the 1-dimensional boundary cycle group, B_1, is written as follows.

$$\begin{aligned} \partial c_2 =& b_1 < (1,2),(2,3) > + (b_2 - b_1) < (1,2),(1,3) > + b_1 < (2,3),(1,3) > \\ &+ b_2 < (3,2),(1,2) > - b_2 < (3,2),(1,3) > \end{aligned}$$

We find that B_1 is isomorphic to Z_1, and so the 1-dimensional homology group is trivial, that is, we have proved $(F \circ i_l)_* = 0$.

(e) Case (v): $\overline{(1,2)} \longrightarrow (2,1),\ \overline{(2,3)} \longrightarrow (2,3)$

The vertices mapped by $F \circ i_l$ to the social preference from the preference of individual l span the following five simplices.

$$< (1,2),(2,3) >,\ < (1,2),(1,3) >,\ < (2,3),(1,3) >,\ < (2,1),(2,3) >,\\ < (2,1),(1,3) >$$

Then, an element of the 1-dimensional chain group is written as

$$\begin{aligned} c_1 =& a_1 < (1,2),(2,3) > + a_2 < (1,2),(1,3) > + a_3 < (2,3),(1,3) > \\ &+ a_4 < (2,1),(2,3) > + a_5 < (2,1),(1,3) > \end{aligned}$$

The condition for an element of the 1-dimensional chain group to be a cycle is

$$\partial c_1 = (-a_1 - a_2) < (1,2) > +(a_1 - a_3 + a_4) < (2,3) > +(a_2 + a_3 + a_5) < (1,3) > + (-a_4 - a_5) < (2,1) >= 0$$

From this

$$-a_1 - a_2 = 0,\ a_1 - a_3 + a_4 = 0,\ a_2 + a_3 + a_5 = 0,\ -a_4 - a_5 = 0$$

are derived, and we obtain $a_2 = -a_1$, $a_5 = -a_4$, $a_3 = a_1 + a_4$. Therefore, an element of the 1-dimensional cycle group is represented as follows.

$$z_1 = a_1 < (1,2),(2,3) > -a_1 < (1,2),(1,3) > +(a_1 + a_4) < (2,3),(1,3) > + a_4 < (2,1),(2,3) > -a_4 < (2,1),(1,3) >$$

On the other hand, the vertices span the following 2-dimensional simplices.

$$< (1,2),(2,3),(1,3) >,\ < (2,1),(2,3),(1,3) >$$

Then, an element of the 2-dimensional chain group is written as

$$c_2 = b_1 < (1,2),(2,3),(1,3) > +b_2 < (2,1),(2,3),(1,3) >$$

And an element of the 1-dimensional boundary cycle group, B_1, is written as follows.

$$\partial c_2 = b_1 < (1,2),(2,3) > -b_1 < (1,2),(1,3) > +(b_1 + b_2) < (2,3),(1,3) > + b_2 < (2,1),(2,3) > -b_2 < (2,1),(1,3) >$$

We find that B_1 is isomorphic to Z_1, and so the 1-dimensional homology group is trivial, that is, we have proved $(F \circ i_l)_* = 0$.

We have completely proved $(F \circ i_l)_* = 0$ in all cases.

□

From these arguments and $(F \circ \Delta)_* \neq 0$ there exists the dictator about x_1, x_2 and x_3. Let individual l be the dictator. Interchanging x_3 with x_4 in the proof of this lemma, we can show that he is the dictator about x_1, x_2 and x_4. Similarly, we can show that he is the dictator about x_5, x_2 and x_4, he is the dictator about x_5, x_6 and x_4. After all he is the dictator about all alternatives, and hence we obtain

Theorem 2.1 (The Arrow impossibility theorem) There exists the dictator for any social welfare function which satisfies transitivity, Pareto principle and IIA.

2.4 Concluding remarks

We have shown the Arrow impossibility theorem when individual preferences are weak orders under the assumption of free-triple property using elementary concepts and techniques of algebraic topology, in particular, homology groups of simplicial complexes and homomorphisms of homology groups induced by simplicial maps.

Our approach may be applied to other problems of social choice theory such as Wilson's impossibility theorem (Wilson (1972)), the Gibbard-Satterthwaite theorem (Gibbard (1973) and Satterthwaite (1975)), and Amartya Sen's liberal paradox (Sen (1979)).

Chapter 3

A topological proof of Eliaz's unified theorem of social choice theory

Recently Eliaz (2004) has presented a unified framework to study (Arrovian) social welfare functions and non-binary social choice functions based on the concept of *preference reversal*. He showed that social choice rules which satisfy the property of preference reversal and a variant of the Pareto principle are dictatorial. This result includes the Arrow impossibility theorem (Arrow (1963)) and the Gibbard-Satterthwaite theorem (Gibbard (1973), Satterthwaite (1975)) as its special cases. We present a concise proof of his theorem using elementary concepts of algebraic topology such as homomorphisms of homology groups of simplicial complexes induced by simplicial mappings[*1].

3.1 Introduction

Recently Eliaz (2004) has presented a unified framework to study (Arrovian) social welfare functions and non-binary social choice functions based on the concept of *preference reversal*. The preference reversal property is a condition (according to the expression in Eliaz (2004)) that if social relation (given by a social choice function or a social preference) between any two alternatives has been reversed, then someone must have exhibited the same reversal in his preference. He showed that social choice rules which satisfy the property of preference reversal and a variant of the Pareto principle are dictatorial. This result includes the Arrow impossibility theorem (Arrow (1963)) and the Gibbard-Satterthwaite theorem (Gibbard (1973), Satterthwaite (1975)) as its special cases. We present a concise proof of his theorem using elementary concepts of algebraic topology such as homomorphisms of homology groups of simplicial complexes induced by simplicial mappings.

Topological approaches to social choice problems have been initiated by Chichilnisky (1980). Her main result is an impossibility theorem that there exists no *continuous* social choice rule which satisfies *unanimity* and *anonymity*. This approach has been further developed by Chichilnisky (1979), (1982), Candeal and Indurain (1994), Koshevoy (1997), Lauwers (2004), Weinberger (2004), and so on. On the other hand, Baryshnikov (1993) and (1997) have presented a topological approach to the Arrow impossibility theorem (or general possibility theorem) in a discrete framework of social choice[*2]. Our research is in line with the

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[*1] This chapter is based on my paper of the same title published in *Applied Mathematics and Computation*, Vol. 176, No. 1, pp. 83-90, 2006, Elsevier.

[*2] About surveys and basic results of topological social choice theories, see Mehta (1997) and Lauwers (2000).

studies of topological approaches to discrete social choice problems initiated by him. In the next section we present expressions of binary social choice rules by simplicial complexes and simplicial mappings. In Section 3.3 we will prove the main results of this chapter.

3.2 The model

There are m alternatives of a social problem, $x_1, x_2, \cdots, x_m$ ($m \geqq 3$), and n individuals ($n \geqq 2$). The set of alternatives is denoted by A. m and n are finite integers. Individual preferences over these alternatives are complete, transitive and asymmetric. Individual i's preference is denoted by P_i. $x_i P_i x_j$ means that he prefers x_i to x_j.

A social choice rule which we will consider according to Eliaz (2004) is a rule that determines a social binary relation about each pair of alternatives corresponding to a combination of individual preferences. It may not be complete. We call such a social choice rule a *binary social choice rule*. It is abbreviated as BCR. We assume the universal (or unrestricted) domain condition for social binary choice rules[*3]. We call a combination of individual preferences a *profile*. The profiles are denoted by $\mathbf{p}$, $\mathbf{p}'$ and so on. Individual i's preference at $\mathbf{p}'$ is denoted by P_i', and so on. A social binary relation generated by a BCR is denoted by R. We call it also a BCR. Let x_i and x_j be two distinct alternatives. $x_i R x_j$ means that x_i *relates to* x_j *according to BCR* R. On the other hand $x_i^{\neg} R x_j$ means that x_i *does not relate to* x_j *according to BCR* R. A BCR at a profile $\mathbf{p}$ is denoted by R, a BCR at $\mathbf{p}'$ is denoted by R', and so on.

Any BCR R is required to satisfy the following conditions.

Existence of a best alternative (BA) There exists an alternative $x_i \in A$ such that $x_i R x_j$ for all $x_j \in A \setminus \{x_i\}$. There may be multiple best alternatives.

Acyclicality (AC) For every three alternatives x_i, x_j and x_k in A if $x_i R x_j$ and $x_k^{\neg} R x_j$, then $x_k^{\neg} R x_i$.

Pareto efficiency (PAR) For every two alternatives x_i and x_j in A if all individuals prefer x_i to x_j, then either "$x_i R x_j$ and $x_j^{\neg} R x_i$", or "x_i and x_j are not related according to R ($x_i^{\neg} R x_j$ and $x_j^{\neg} R x_i$)".

Preference reversal (PR) For every two alternatives x_i and x_j in A if $x_i R x_j$, $x_j^{\neg} R x_i$ but $x_j R' x_i$, then there exists (at least) one individual i such that $x_i P_i x_j$ and $x_j P_i' x_i$.

Dictator is defined as follows.

Dictator If, there exists an individual i such that for every pair of alternatives x_i and x_j the social relation is $x_j^{\neg} R x_i$ whenever he prefers x_i to x_j, then he is the dictator of R.

As proved in Observation 1 of Eliaz (2004) AC is equivalent to the following Transitivity.

Transitivity (T) For every three alternatives x_i, x_j and x_k in A if $x_i R x_j$ and $x_j R x_k$, then $x_i R x_k$.

Proof. 1. AC$\longrightarrow$T: Assume that $x_i R x_j$, $x_j R x_k$ but $x_i^{\neg} R x_k$. Then, from $x_j R x_k$ and $x_i^{\neg} R x_k$ AC implies $x_i^{\neg} R x_j$. It is a contradiction.

2. T$\longrightarrow$AC: Assume that $x_i R x_j$, $x_k^{\neg} R x_j$ but $x_k R x_i$. Then, from $x_k R x_i$ and $x_i R x_j$ T implies $x_k R x_j$. It is a contradiction.

□

As noted by Eliaz (2004) if a BCR satisfies BA, AC and the Completeness (Condition C) ($x_i R x_j$ or

[*3] The universal domain condition means that the domain of individuals preferences for social binary choice rules is never restricted.

$x_j Rx_i$), then it is an Arrovian social welfare function. In this interpretation AC means the transitivity of strict social preferences[*4]. Eliaz (2004) showed that if a social welfare function satisfies BA, AC, PAR, C and Arrow's condition of *independence of irrelevant alternatives*, then it satisfies PR. If a BCR satisfies C, $x_j^{\neg} Rx_i$ is equivalent to $x_i Rx_j$. Thus, the dictator in the above definition is the dictator for an Arrovian social welfare function.

On the other hand, if the unique alternative x_i satisfies $x_i Rx_j$ for all $x_j \in A \setminus \{x_i\}$ and all alternatives other than x_i are not mutually related according to a BCR R, then it is a social choice function which is a social choice rule that chooses one alternative corresponding to each profile. To be precise a social choice function chooses one alternative corresponding to a profile of *reported preferences* of individuals. If a social choice function does not give any incentive to every individual to report a preference which is different from his true preference, then it is *strategy-proof*. It was shown by Eliaz (2004) that a strategy-proof social choice function satisfies PR. If there exists the unique best alternative x_i for a BCR, then $x_j^{\neg} Rx_i$ means that x_j is not chosen by the social choice function derived from this BCR, and the dictator in the above definition is the dictator for the social choice function. Eliaz (2004) showed the theorem that if a BCR satisfies BA, AC, PAR and PR, it has the dictator. Then, the Arrow impossibility theorem that there exists the dictator for any social welfare function which satisfies BA, AC, C, PAR and the independence of irrelevant alternatives under the universal domain condition, and the Gibbard-Satterthwaite theorem that there exists the dictator for any social choice function which is onto (surjection) and strategy-proof under the universal domain condition are the special cases of his theorem.

PAR with BA implies the following condition[*5].

Strong Pareto efficiency (SPAR) For every alternative x_i if all individuals prefer x_i to all other alternatives, then we have $x_i Rx_j$ and $x_j^{\neg} Rx_i$ for all $x_j \in A \setminus \{x_i\}$.

Now we consider topological expressions of individual preferences. We draw a circumference which represents the set of individual preferences by connecting $m!$ vertices $v_1, v_2, \cdots, v_{m!}$ by arcs[*6]. For example, in the case of four alternatives, these vertices mean the following preferences.

$$v_1 : (1234),\ v_2 : (1243)\ v_3 : (1423),\ v_4 : (1432),\ v_5 : (1342),\ v_6 : (1324)$$

$$v_7 : (2134),\ v_8 : (2143)\ v_9 : (2413),\ v_{10} : (2431),\ v_{11} : (2341),\ v_{12} : (2314)$$

$$v_{13} : (3124),\ v_{14} : (3142)\ v_{15} : (3412)\ v_{16} : (3421),\ v_{17} : (3241),\ v_{18} : (3214)$$

$$v_{19} : (4123),\ v_{20} : (4132)\ v_{21} : (4312)\ v_{22} : (4321),\ v_{23} : (4231),\ v_{24} : (4213)$$

We denote a preference such that an individual prefers x_1 to x_2 to x_3 to x_4 by (1234), and so on. Notations for the cases with different number of alternatives are similar. Generally $v_1 \sim v_{(m-1)!}$ represent preferences such that the most preferred alternative for an individual is x_1, $v_{(m-1)!+1} \sim v_{2(m-1)!}$ represent preferences

*4 From Lemma 1 of Baryshnikov (1993) we know that if individual preferences are strict orders, then the social preference is also a strict order under the transitivity, the Pareto principle and the independence of irrelevant alternatives.

*5 This term SPAR is not defined in Eliaz (2004).

*6 $m!$ denotes factorial of m.

$$m! = \prod_{j=1}^{m} j = m(m-1)(m-2) \times \cdots \times 2 \times 1$$

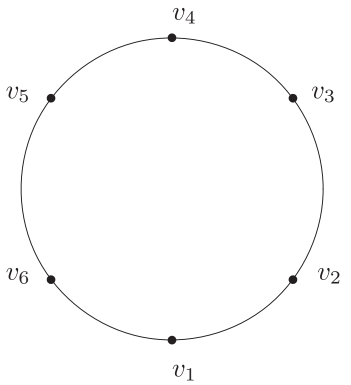


Figure 1: S_i^1

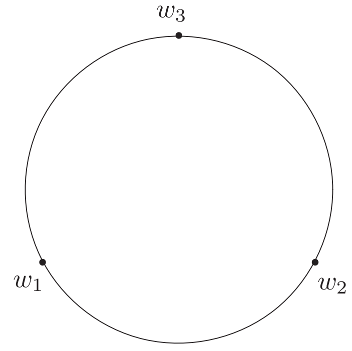


Figure 2: S^1

such that the most preferred alternative for an individual is x_2, and so on. In particular v_1 denotes a preference such that an individual prefers x_1 to x_2 to x_3 to $\cdots$ to x_m. It is denoted by $(123\cdots m)$.

Denote this circumference by S_i^1. S_i^1 in the case of three alternatives is depicted in Figure 1. The set of profiles of the preferences of n individuals is represented by the product space $S_i^1 \times \cdots \times S_i^1$ (n times). It is denoted by $(S_i^1)^n$. The 1-dimensional homology group of S_i^1 is isomorphic to the group of integers $\mathbb{Z}$, that is, $H_1(S_i^1) \cong \mathbb{Z}$. And the 1-dimensional homology group of $(S_i^1)^n$ is isomorphic to the direct product of n groups of integers $\mathbb{Z}^n$, that is, we have $H_1((S_i^1)^n) \cong \mathbb{Z}^n$. It is proved, for example, using the Mayer-Vietoris exact sequences*7.

The social binary relation generated by a BCR is also represented by a circumference depicted in Figure 2. This circumference is drawn by connecting three vertices, w_1, w_2 and w_3 by arcs. These vertices mean the following social binary relations.

1. w_2: binary relations such that $x_2 R x_j$ and $x_j \neg R x_2$ for all $x_j \in A \setminus \{x_2\}$.
2. w_3: binary relations such that $x_3 R x_j$ and $x_j \neg R x_3$ for all $x_j \in A \setminus \{x_3\}$.
3. w_1: all other social binary relations.

We call this circumference S^1. The 1-dimensional homology group of S^1 is also isomorphic to $\mathbb{Z}$, that is, $H_1(S^1) \cong \mathbb{Z}$.

Binary social choice rules are simplicial mappings. Binary social choice rules are denoted by $f : (S_i^1)^n \longrightarrow S^1$. Two adjacent vertices of S_i^1 span a 1-dimensional simplex. And any pair of two vertices of S^1 spans a 1-dimensional simplex. Thus, f is a simplicial mapping, and we can define the homomorphism of homology groups induced by f.

We define an inclusion mapping from S_i^1 to $(S_i^1)^n$ by $\Delta : S_i^1 \longrightarrow (S_i^1)^n$ under the assumption that all individuals have the same preferences, and define an inclusion mapping when the profile of preferences of individuals other than one individual (denoted by i) is fixed at some profile by $i_i : S_i^1 \longrightarrow (S_i^1)^n$. The

*7 About homology groups and the Mayer-Vietoris exact sequences we referred to Tamura (1970) and Komiya (2001).

homomorphisms of homology groups induced by these inclusion mappings are as follows.

$$\Delta_* : \mathbb{Z} \longrightarrow \mathbb{Z}^n : h \longrightarrow (h, h, \cdots, h), \ h \in \mathbb{Z}$$

$$i_{i*} : \mathbb{Z} \longrightarrow \mathbb{Z}^n : h \longrightarrow (0, \cdots, 0, h, 0, \cdots, 0), \ h \in \mathbb{Z} \text{ (only the } i\text{-th component is } h)$$

From these definitions we obtain the following relation about Δ_* and i_{i*} at any profile.

$$\Delta_* = \sum_{i=1}^{n} i_{i*} \tag{3.1}$$

Let us denote the homomorphism of homology groups induced by f by $f_* : (\mathbb{Z})^n \longrightarrow \mathbb{Z}$.

Binary social choice rules for different profiles are homotopic. f for a fixed profile of preferences of individuals other than i (denoted by $f|_{\mathbf{p}_{-i}}$) and f for another fixed profile of their preferences (denoted by $f|_{\mathbf{p}'_{-i}}$) are homotopic. Thus, the homomorphisms of homology groups induced by them are isomorphic. Denote two profiles of individuals other than i by $\mathbf{p}_{-i}$ and $\mathbf{p}'_{-i}$. Then, the homotopy between $f|_{\mathbf{p}_{-i}}$ and $f|_{\mathbf{p}'_{-i}}$ is

$$f_t = \frac{tf|_{\mathbf{p}_{-i}} + (1-t)f|_{\mathbf{p}'_{-i}}}{|tf|_{\mathbf{p}_{-i}} + (1-t)f|_{\mathbf{p}'_{-i}}|} \ (0 \leqq t \leqq 1)$$

It is well defined since $f|_{\mathbf{p}_{-i}}$ and $f|_{\mathbf{p}'_{-i}}$ are not anti-podal.

The composite function of i_i and f is denoted by $f \circ i_i : S^1_i \longrightarrow S^1$, and its induced homomorphism of homology groups satisfies $(f \circ i_i)_* = f_* \circ i_{i*}$, for all i. The composite function of Δ and f is denoted by $f \circ \Delta : S^1_i \longrightarrow S^1$, and its induced homomorphism of homology groups satisfies $(f \circ \Delta)_* = f_* \circ \Delta_*$. From (3.1) we obtain

$$(f \circ \Delta)_* = \sum_{i=1}^{n} (f \circ i_i)_* \tag{3.2}$$

3.3 The main results

In this section we will prove the following theorem by Eliaz (2004).

Theorem 3.1 There exists the dictator for any BCR which satisfies BA, AC, PAR and PR.

First we show the following lemma which will be used below.

Lemma 3.1 Suppose that a BCR satisfies BA, AC, PAR and PR, and has no dictator. When the preference of one individual (denoted by i) is $(234 \cdots m1)$, and the preferences of all other individuals are v_1, then we have

$x_1 R x_j$ and $x_j^{\neg} R x_1$ for all $x_j \in A \setminus \{x_1, x_2\}$

Proof. Step 1:

Note that v_1 represents a preference $(123 \cdots m)$. By PAR we have

$$x_2 R x_j \text{ (or } x_2^{\neg} R x_j) \text{ and } x_j^{\neg} R x_2 \text{ for all } x_j \in A \setminus \{x_1, x_2\} \tag{3.3}$$

By BA there are the following three cases about x_1 and x_2[*8].

1. Case 1: $x_2 R x_1$ and $x_1 \neg R x_2$.
2. Case 2: $x_1 R x_2$ and $x_2 \neg R x_1$.
3. Case 3: $x_1 R x_2$ and $x_2 R x_1$.

It will be proved that in Case 1 individual i is the dictator. In Step 1 we consider this case. By PR we have $x_1 \neg R x_2$ so long as individual i prefers x_2 to x_1. Then, we say that individual i is *decisive* for x_2 against x_1. Let x_j and x_k ($x_k \neq x_j$) be alternatives other than x_1 and x_2, and consider the following profile.

1. Individual i prefers x_k to x_2 to x_1 to x_j to all other alternatives.
2. Other individuals prefer x_1 to x_j to x_k to x_2 to all other alternatives.

By PR we have $x_1 \neg R x_2$. And by PAR we have

1. $x_1 R x_j$ (or $x_1 \neg R x_j$) and $x_j \neg R x_1$, and $x_1 R x_l$ (or $x_1 \neg R x_l$) and $x_l \neg R x_1$ for all $x_l \in A \setminus \{x_1, x_2, x_j, x_k\}$.
2. $x_k R x_2$ (or $x_k \neg R x_2$) and $x_2 \neg R x_k$, and $x_k R x_l$ (or $x_k \neg R x_l$) and $x_l \neg R x_k$ for all $x_l \in A \setminus \{x_1, x_2, x_j, x_k\}$.

BA and AC imply that we have $x_k R x_l$ and $x_l \neg R x_k$ for all $x_l \in A \setminus \{x_k\}$[*9]. Then, by PR we have $x_j \neg R x_k$ so long as individual i prefers x_k to x_j, and so individual i is decisive for x_k against x_j. Note that x_j and x_k are arbitrary. Next consider the following profile.

1. Individual i prefers x_2 to x_k to x_j to all other alternatives.
2. Other individuals prefer x_j to x_2 to x_k to all other alternatives.

By PR we have $x_j \neg R x_k$. And by PAR we have

$$x_2 R x_k \text{ (or } x_2 \neg R x_k\text{) and } x_k \neg R x_2\text{, and } x_2 R x_l \text{ (or } x_2 \neg R x_l\text{) and } x_l \neg R x_2 \text{ for all } x_l \in A \setminus \{x_2, x_j, x_k\}.$$

BA and AC imply that we have $x_2 R x_l$ and $x_l \neg R x_2$ for all $x_l \in A \setminus \{x_2\}$. Then, by PR we have $x_j \neg R x_2$ so long as individual i prefers x_2 to x_j, and so individual i is decisive for x_2 against x_j. Next consider the following profile.

1. Individual i prefers x_k to x_j to x_2 to all other alternatives.
2. Other individuals prefer x_j to x_2 to x_k to all other alternatives.

By PR we have $x_j \neg R x_k$. And by PAR we have

$$x_j R x_2 \text{ (or } x_j \neg R x_2\text{) and } x_2 \neg R x_j\text{, and } x_j R x_l \text{ (or } x_j \neg R x_l\text{) and } x_l \neg R x_j \text{ for all } x_l \in A \setminus \{x_2, x_j, x_k\}.$$

BA and AC imply that we have $x_k R x_l$ and $x_l \neg R x_k$ for all $x_l \in A \setminus \{x_k\}$. Then, by PR we have $x_2 \neg R x_k$ so long as individual i prefers x_k to x_2, and so individual i is decisive for x_k against x_2. By similar procedures we can show that individual i is decisive for x_1 against x_j, and is decisive for x_k against x_1. Finally consider the following profile.

1. Individual i prefers x_1 to x_k to x_2 to all other alternatives.
2. Other individuals prefer x_2 to x_1 to x_k to all other alternatives.

[*8] If $x_1 \neg R x_2$ and $x_2 \neg R x_1$, then there exists no best alternative.

[*9] BA implies $x_k R x_l$ for all $x_l \in A \setminus \{x_k\}$, and from AC with $x_1 \neg R x_2$, $x_j \neg R x_1$, $x_2 \neg R x_k$ and $x_l \neg R x_k$ ($x_l \in A \setminus \{x_1, x_2, x_j, x_k\}$) we have $x_l \neg R x_k$ for all $x_l \in A \setminus \{x_k\}$.

By PR we have $x_2^{-}Rx_k$. And by PAR we have

$$x_1Rx_k \text{ (or } x_1^{-}Rx_k\text{) and } x_k^{-}Rx_1, \text{ and } x_1Rx_l \text{ (or } x_1^{-}Rx_l\text{) and } x_l^{-}Rx_1 \text{ for all } x_l \in A \setminus \{x_1, x_2, x_k\}.$$

BA and AC imply that we have x_1Rx_l and $x_l^{-}Rx_1$ for all $x_l \in A \setminus \{x_1\}$. Then, by PR we have $x_2^{-}Rx_1$ so long as individual i prefers x_1 to x_2, and individual i is decisive for x_1 against x_2. Therefore, individual i is the dictator*10.

Step 2:

Next let us consider Case 2 and 3. From (3.3) we have $x_j^{-}Rx_2$ for all $x_j \in A \setminus \{x_1, x_2\}$. Then in both Case 2 and 3, x_1Rx_2 and AC imply

$$x_j^{-}Rx_1 \text{ for all } x_j \in A \setminus \{x_1, x_2\}$$

By BA in Case 2 we obtain

$$x_1Rx_j \text{ and } x_j^{-}Rx_1 \text{ for all } x_j \in A \setminus \{x_1\}.$$

And in Case 3 we have*11

$$x_1Rx_2,\ x_2Rx_1,\ x_1Rx_j \text{ and } x_j^{-}Rx_1 \text{ for all } x_j \in A \setminus \{x_1, x_2\}. \tag{3.4}$$

Therefore, we get the conclusion of this lemma. □

By SPAR we obtain the correspondences from the vertices of S_i^1 to the vertices of S^1 by $f \circ \Delta$ as follows.

$$v_1 \sim v_{(m-1)!} \longrightarrow w_1,\ v_{(m-1)!+1} \sim v_{2(m-1)!} \longrightarrow w_2,\ v_{2(m-1)!+1} \sim v_{3(m-1)!} \longrightarrow w_3$$

All other vertices correspond to w_1. Sets of 1-dimensional simplices included in S_i^1 which are 1-dimensional cycles are only the following z and its counterpart $-z$.

$$z = < v_1, v_2 > + < v_2, v_3 > + \cdots + < v_{m!-1}, v_{m!} > + < v_{m!}, v_1 >$$

Since S_i^1 does not have a 2-dimensional simplex, z is a representative element of homology classes of S_i^1. z is transferred by $(f \circ \Delta)_*$ to the following z'.

$$z' = < w_1, w_2 > + < w_2, w_3 > + < w_3, w_1 >$$

This is a cycle of S^1. Therefore, we have

$$(f \circ \Delta)_* \neq 0 \tag{3.5}$$

Now we show the following lemma.

Lemma 3.2 If a BCR satisfies BA, AC, PAR and PR, and has no dictator, then we obtain

$$(f \circ i_i)_* = 0 \text{ for all } i \tag{3.6}$$

*10 We can show that individual i is the dictator in Case 1 when there are only three alternatives by similar procedures.

*11 By BA we obtain

$$x_1Rx_j \text{ for all } x_j \in A \setminus \{x_1\}, \text{ or } x_2Rx_j \text{ for all } x_j \in A \setminus \{x_2\}$$

Then, AC or T(Transitivity) implies (3.4).

Proof. By SPAR when the preference of every individual other than one individual (denoted by i) is fixed at v_1, the correspondences from the preference of individual i to the social binary relation from v_1 to $v_{(m-1)!}$ are as follows.

$$v_1 \sim v_{(m-1)!} \longrightarrow w_1$$

Lemma 3.1 implies that the correspondence from $(234\cdots m1)$ to the social binary relation is as follows.

$$(234\cdots m1) \longrightarrow w_1, \text{ and we have } x_1 R x_j,\ x_j \neg R x_1 \text{ for all } x_j \in A \setminus \{x_1, x_2\}$$

Then, PR implies that x_3 is never the unique best alternative for BCR so long as the most preferred alternative for all individuals other than i is x_1 regardless of the preference of individual i, and so the preference of individual i corresponds to w_1 or w_2. Thus, we obtain the following correspondences.

$$v_{(m-1)!+1} \sim v_{m!} \longrightarrow w_1 \text{ or } w_2$$

Sets of 1-dimensional simplices included in S_i^1 which are 1-dimensional cycles are only the following z and its counterpart $-z$.

$$z = < v_1, v_2 > + < v_2, v_3 > + \cdots + < v_{m!-1}, v_{m!} > + < v_{m!}, v_1 >$$

Since S_i^1 does not have a 2-dimensional simplex, z is a representative element of homology classes of S_i^1. z is transferred by $(f \circ i_l)_*$ to the following z'.

$$z' =< w_1, w_2 > + < w_2, w_1 >= 0 \text{ or } z' =< w_1, w_1 >= 0$$

Therefore, we have $(f \circ i_i)_* = 0$ for all i. □

The conclusion of this lemma contradicts (3.2) and (3.5). Therefore, we have shown Theorem 3.1. We call the property expressed in (3.6) the *non-surjectivity of individual inclusion mappings*. Then, Theorem 3.1 is a special case of the following theorem.

Theorem 3.2 There exists no binary social choice rule which satisfies SPAR and the non-surjectivity of individual inclusion mappings.

From (3.5) SPAR implies the *surjectivity of the diagonal mapping*, $(f \circ \Delta)_* \neq 0$, for binary social choice rules. Thus, this theorem is rewritten as follows.

> There exists no binary social choice rule which satisfies the *surjectivity of the diagonal mapping* and the *non-surjectivity of individual inclusion mappings*.

3.4 Concluding remarks

In Baryshnikov (1997) he said, "the similarities between the two theories, the classical and topological ones, are somewhat more extended than one would expect. The details seem to fit too well to represent just an analogy. I would conjecture that the homological way of proving results in both theories is a 'true' one because of its uniformity and thus can lead to much deeper understanding of the structure of social choice. To understand this structure better we need a much more evolved collection of examples of unifying these two theories and I hope this can and will be done." This chapter is an attempt to provide such an example.

Chapter 4

On the topological equivalence of the Arrow impossibility theorem and Amartya Sen's liberal paradox

We will show that the Arrow impossibility theorem for binary social choice rules that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives (IIA), and has no dictator, and Amartya Sen's liberal paradox for binary social choice rules that there exists no binary social choice rule which satisfies acyclicity, Pareto principle and the minimal liberalism are topologically equivalent using elementary tools of algebraic topology such as homomorphisms of homology groups of simplicial complexes induced by simplicial mappings. Our research is in line with the studies of topological approaches to discrete social choice problems initiated by Baryshnikov (1993). Also we will show that these two theorems are special cases of the theorem that there exists no binary social choice rule which satisfies Pareto principle and the *non-surjectivity of individual inclusion mappings*[*1].

4.1 Introduction

Topological approaches to social choice problems have been initiated by Chichilnisky (1980). Her main result is an impossibility theorem that there exists no *continuous* social choice rule which satisfies *unanimity* and *anonymity*. This approach has been further developed by Chichilnisky (1979), (1982), Candeal and Indurain (1994), Koshevoy (1997), Lauwers (2004), Weinberger (2004), and so on. On the other hand, Baryshnikov (1993) and (1997) have presented a topological approach to the Arrow impossibility theorem (or general possibility theorem) in a discrete framework of social choice[*2].

We will show that the Arrow impossibility theorem for binary social choice rules that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives (IIA), and has no dictator, and Amartya Sen's liberal paradox for binary social choice rules that there exists no binary social choice rule which satisfies acyclicity, Pareto principle and the minimal liberalism are topologically equivalent using elementary tools of algebraic topology such as homomorphisms

[*1] This chapter is based on my paper of the same title published in *Applied Mathematics and Computation*, Vol. 181, No. 2, pp. 1490-1498, 2006, Elsevier.

[*2] About surveys and basic results of topological social choice theories, see Mehta (1997) and Lauwers (2000).

of homology groups of simplicial complexes induced by simplicial mappings. Also we will show that these two theorems are special cases of the theorem that there exists no binary social choice rule which satisfies Pareto principle and the *non-surjectivity of individual inclusion mappings*. Our research is in line with the studies of topological approaches to discrete social choice problems initiated by Baryshnikov (1993).

In the next section we present expressions of binary social choice rules by simplicial complexes and simplicial mappings. In Section 4.3 we will prove the main results of this chapter.

4.2 The expressions of social choice problems by simplicial complexes and simplicial mappings

There are m alternatives of a social problem, $x_1, x_2, \cdots, x_m$ ($m \geqq 3$), and n individuals ($n \geqq 2$). m and n are finite integers. Individual preferences over these alternatives are complete, transitive and asymmetric.

A social choice rule which we will consider is a rule that determines a social preference about each pair of alternatives corresponding to a combination of individual preferences. We call such a social choice rule a *binary social choice rule*. The social preference should be complete, but may be or may not be transitive. As usual we assume the universal domain condition for social choice rules. We call a combination of individual preferences a *profile*. The profiles are denoted by $\mathbf{p}$, $\mathbf{p}'$ and so on.

We will consider two social choice problems about binary social choice rules.

1. **(Amartya Sen's liberal paradox)** The liberal paradox by Amartya Sen (Sen (1979)) states that there exists no binary social choice rule which satisfies *acyclicity*, *Pareto principle* and the *minimal liberalism*. The means of these conditions are as follows.

   Acyclicity If the society (strictly) prefers x_i to x_j, and (strictly) prefers x_j to x_k, then it should prefer x_i to x_k or be indifferent between them. It is weaker than transitivity which requires that the society (strictly) prefers x_i to x_k.

   Pareto principle If all individuals prefer an alternative x_i to another alternative x_j, then the society prefers x_i to x_j.

   Minimal liberalism At least two individuals, denoted by A and B, are decisive for some pairs of alternatives in both directions in the sense described in the following Assumption 1.

   In what follows as the condition of the minimal liberalism we assume

   Assumption 1 If individual A prefers x_1 to x_3 (or prefers x_3 to x_1), then the society prefers x_1 to x_3 (or prefers x_3 to x_1). And if individual B prefers x_2 to x_4 (or prefers x_4 to x_2), then the society prefers x_2 to x_4 (or prefers x_4 to x_2).

   Other individuals are not necessarily decisive. We can proceed the arguments in a similar manner based on other assumptions about the minimal liberalism by permuting or renaming alternatives. We abbreviate the problem of the liberal paradox as LP.

2. **(The Arrow impossibility theorem)** The Arrow impossibility theorem (Arrow (1963)) states that there exists no binary social choice rule which satisfies *transitivity*, *Pareto principle* and *independence of irrelevant alternatives (IIA)*, and *has no dictator*, or in other words there exists the dictator for any binary social choice rule which satisfies transitivity, Pareto principle and IIA. The dictator for a binary social choice rule is an individual such that whenever he (strictly) prefers one alternative (denoted by x) to another alternative (denoted by y), the society also (strictly) prefers x to y. The

meanings of two conditions, transitivity and IIA, are as follows.

Transitivity If the society prefers x_i to x_j, and prefers x_j to x_k, then the society should prefer x_i to x_k.

Independence of irrelevant alternatives (IIA) The society's preference about any pair of two alternatives depends only on individual preferences about these alternatives.

We abbreviate the problem of the Arrow impossibility theorem as AR. Pareto principle for AR is the same as that for LP.

We draw a circumference which represents the set of individual preferences by connecting $m!$ vertices $v_1, v_2, \cdots, v_{m!}$ by arcs*3. For example, in the case of four alternatives, these vertices mean the following preferences.

$$v_1 : (1234),\ v_2 : (1243)\ v_3 : (1423),\ v_4 : (1432),\ v_5 : (1342),\ v_6 : (1324)$$

$$v_7 : (2134),\ v_8 : (2143)\ v_9 : (2413),\ v_{10} : (2431),\ v_{11} : (2341),\ v_{12} : (2314)$$

$$v_{13} : (3124),\ v_{14} : (3142)\ v_{15} : (3412)\ v_{16} : (3421),\ v_{17} : (3241),\ v_{18} : (3214)$$

$$v_{19} : (4123),\ v_{20} : (4132)\ v_{21} : (4312)\ v_{22} : (4321),\ v_{23} : (4231),\ v_{24} : (4213)$$

We denote a preference such that an individual prefers x_1 to x_2 to x_3 to x_4 by (1234), and so on. Notations for the cases with different number of alternatives are similar. Generally $v_1 \sim v_{(m-1)!}$ represent preferences such that the most preferred alternative for an individual is x_1, $v_{(m-1)!+1} \sim v_{2(m-1)!}$ represent preferences such that the most preferred alternative for an individual is x_2, and so on. And v_1 is a preference such that an individual prefers x_1 to x_2 to x_3 to $\cdots$ to x_m. It is denoted by $(123\cdots m)$. $v_{(m-1)!+1}$ is a preference such that an individual prefers x_2 to x_1 to x_3 to x_4 to $\cdots$ to x_m, which is denoted by $(2134\cdots m)$.

Denote this circumference by S_i^1. S_i^1 in the case of three alternatives is depicted in Figure 1. The set of profiles of the preferences of n individuals is represented by the product space $S_i^1 \times \cdots \times S_i^1$ (n times). It is denoted by $(S_i^1)^n$. The 1-dimensional homology group of S_i^1 is isomorphic to the group of integers $\mathbb{Z}$, that is, $H_1(S_i^1) \cong \mathbb{Z}$. And the 1-dimensional homology group of $(S_i^1)^n$ is isomorphic to the direct product of n groups of integers $\mathbb{Z}^n$, that is, we have $H_1((S_i^1)^n) \cong \mathbb{Z}^n$. It is proved, for example, using the Mayer-Vietoris exact sequences*4.

The social preference is also represented by a circumference depicted in Figure 2. This circumference is drawn by connecting three vertices, w_1, w_2 and w_3 by arcs. For LP these vertices mean the following social preferences.

1. w_1: social preferences such that the society prefers x_4 to all other alternatives.
2. w_3: social preferences such that the society prefers x_3 to all other alternatives.
3. w_2: all other social preferences.

Similarly, for AR these vertices mean the following social preferences.

*3 $m!$ denotes factorial of m.

$$m! = \prod_{j=1}^{m} j = m(m-1)(m-2) \times \cdots \times 2 \times 1$$

*4 About homology groups and the Mayer-Vietoris exact sequences we referred to Tamura (1970) and Komiya (2001).

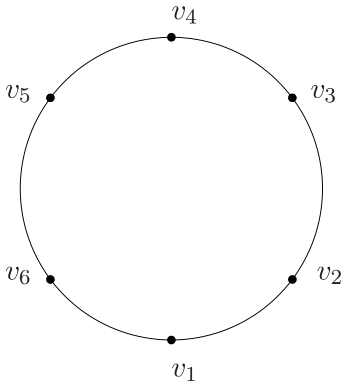


Figure 1: S_i^1

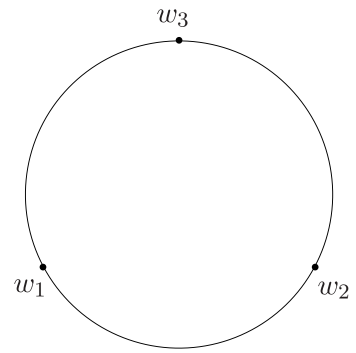


Figure 2: S^1

1. w_1: social preferences such that the society prefers x_4 to all other alternatives.
2. w_3: social preferences such that the society prefers x_3 to all other alternatives.
3. w_2: all other social preferences.

That is, the vertices w_1 and w_3 denote the same social preferences for LP and AR, and the set of social preferences expressed by w_2 for AR is the proper subset of the set of social preferences expressed by w_2 for LP because the social preference are required to satisfy transitivity in AR, but in LP we require only acyclicity.

We call this circumference S^1. The 1-dimensional homology group of S^1 is also isomorphic to $\mathbb{Z}$, that is, $H_1(S^1) \cong \mathbb{Z}$.

Binary social choice rules are simplicial mappings. Binary social choice rules in AR and LP are denoted by $f : (S_i^1)^n \longrightarrow S^1$. Two adjacent vertices of S_i^1 span a 1-dimensional simplex. And any pair of two vertices of S^1 spans a 1-dimensional simplex. Thus, f is a simplicial mapping, and we can define the homomorphism of homology groups induced by f.

We define an inclusion mapping from S_i^1 to $(S_i^1)^n$ by $\Delta : S_i^1 \longrightarrow (S_i^1)^n$ under the assumption that all individuals have the same preferences, and define an inclusion mapping when the profile of preferences of individuals other than one individual (denoted by i) is fixed at some profile by $i_i : S_i^1 \longrightarrow (S_i^1)^n$. The homomorphisms of homology groups induced by these inclusion mappings are as follows.

$$\Delta_* : \mathbb{Z} \longrightarrow \mathbb{Z}^n : h \longrightarrow (h, h, \cdots, h), \ h \in \mathbb{Z}$$

$$i_{i*} : \mathbb{Z} \longrightarrow \mathbb{Z}^n : h \longrightarrow (0, \cdots, 0, h, 0, \cdots, 0), \ h \in \mathbb{Z} \text{ (only the } i\text{-th component is } h)$$

From these definitions we obtain the following relation about Δ_* and i_{i*} at any profile.

$$\Delta_* = \sum_{i=1}^{n} i_{i*} \tag{4.1}$$

Let the homomorphism of homology groups induced by f be $f_* : (\mathbb{Z})^n \longrightarrow \mathbb{Z}$.

Binary social choice rules for different profiles are homotopic. f for a fixed profile of the preferences of individuals other than i (denoted by $f|_{\mathbf{p}_{-i}}$) and f for another fixed profile of the preferences of individuals other than i (denoted by $f|_{\mathbf{p}'_{-i}}$) are homotopic. Thus, the homomorphisms of homology groups induced by them are isomorphic. Denote two profiles of individuals other than i by $\mathbf{p}_{-i}$ and $\mathbf{p}'_{-i}$. Then, the homotopy between $f|_{\mathbf{p}_{-i}}$ and $f|_{\mathbf{p}'_{-i}}$ is $f_t = \frac{tf|_{\mathbf{p}_{-i}}+(1-t)f|_{\mathbf{p}'_{-i}}}{|tf|_{\mathbf{p}_{-i}}+(1-t)f|_{\mathbf{p}'_{-i}}|}$ $(0 \leqq t \leqq 1)$. It is well defined since $f|_{\mathbf{p}_{-i}}$ and $f|_{\mathbf{p}'_{-i}}$ are not anti-podal.

The composite function of i_i and f is denoted by $f \circ i_i : S_i^1 \longrightarrow S^1$, and its induced homomorphism of homology groups satisfies $(f \circ i_i)_* = f_* \circ i_{i*}$, for all i. The composite function of Δ and f is denoted by $f \circ \Delta : S_i^1 \longrightarrow S^1$, and its induced homomorphism of homology groups satisfies $(f \circ \Delta)_* = f_* \circ \Delta_*$. From (4.1) we obtain

$$(f \circ \Delta)_* = \sum_{i=1}^{n} (f \circ i_i)_* \tag{4.2}$$

4.3 The main results

For binary social choice rules in AR we define the following concept.

Weak monotonicity For two alternatives x_i and x_j, suppose that at profile p the society prefers x_i to x_j. And suppose that individuals, who prefer x_i to x_j at p, prefer x_i to x_j at another profile p'. Then, the society prefers x_i to x_j at p'.

We show the following result.

Lemma 4.1 Any binary social choice rule in AR which satisfies transitivity, Pareto principle and IIA satisfies the weak monotonicity.

Proof. We use notations in the definition of the weak monotonicity. Let x_k be an arbitrary alternative other than x_i and x_j.

Suppose that individuals, who prefer x_i to x_j at p, prefer x_i to x_j to x_k at another profile p'', and individuals, who prefer x_j to x_i at p, prefer x_j to x_k to x_i at p''.

And suppose that individuals, who prefer x_i to x_j at p, prefer x_i to x_k to x_j at another profile p^*, and individuals, who prefer x_j to x_i at p, prefer x_k to x_i and prefer x_k to x_j at p^* (their preferences about x_i and x_j are not specified).

By transitivity, Pareto principle and IIA the society prefers x_i to x_j to x_k at p''. Again by transitivity, Pareto principle and IIA (about x_i and x_k) the society prefers x_i to x_k to x_j at p^*. Then, IIA implies that the society prefers x_i to x_j so long as individuals, who prefer x_i to x_j at p, prefer x_i to x_j at an arbitrary profile p'. □

Next we show the following lemma which will be used below.

Lemma 4.2 Suppose that a binary social choice rule satisfies transitivity, Pareto principle, IIA, and has no dictator. If the preference of one individual (denoted by i) is $v_{(m-1)!+1}$, and the preferences of all other individuals are v_1, then the most preferred alternative for the society is x_1.

Proof. Note that $v_{(m-1)!+1}$ represents a preference $(2134 \cdots m)$, and v_1 represents a preference $(123 \cdots m)$. By Pareto principle the society prefers x_1 and x_2 to all other alternatives. It may prefer x_1 to x_2, or x_2

to x_1[*5]. But we can show that if the society prefers x_2 to x_1, individual i is the dictator. Assume that the society prefers x_2 to x_1 to all other alternatives. By the weak monotonicity the society prefers x_2 to x_1 so long as individual i prefers x_2 to x_1. Then, we say that individual i is *decisive* for x_2 against x_1. Let x_j and x_k ($x_k \neq x_j$) be alternatives other than x_1 and x_2, and consider the following profile.

1. Individual i prefers x_k to x_2 to x_1 to x_j.
2. Other individuals prefer x_1 to x_j to x_k to x_2.

By the weak monotonicity (or IIA) the society should prefer x_2 to x_1. And by Pareto principle the society should prefer x_1 to x_j, and prefer x_k to x_2. Then, transitivity implies that the society prefers x_k to x_j. The weak monotonicity implies that the society prefers x_k to x_j so long as individual i prefers x_k to x_j, and individual i is decisive for x_k against x_j. Note that x_j and x_k are arbitrary. Next consider the following profile.

1. Individual i prefers x_2 to x_k to x_j.
2. Other individuals prefer x_j to x_2 to x_k.

By the weak monotonicity (or IIA) the society should prefer x_k to x_j. And by Pareto principle the society should prefer x_2 to x_k. Then, transitivity implies that the society prefers x_2 to x_j. The weak monotonicity implies that the society prefers x_2 to x_j so long as individual i prefers x_2 to x_j, and individual i is decisive for x_2 against x_j. Consider the following profile.

1. Individual i prefers x_k to x_j to x_2.
2. Other individuals prefer x_j to x_2 to x_k.

By the weak monotonicity (or IIA) the society should prefer x_k to x_j. And by Pareto principle the society should prefer x_j to x_2. Then, transitivity implies that the society prefers x_k to x_2. The weak monotonicity implies that the society prefers x_k to x_2 so long as individual i prefers x_k to x_2, and individual i is decisive for x_k against x_2. By similar procedures we can show that individual i is decisive for x_1 against x_j, and is decisive for x_k against x_1. Finally consider the following profile.

1. Individual i prefers x_1 to x_k to x_2.
2. Other individuals prefer x_2 to x_1 to x_k.

By the weak monotonicity (or IIA) the society should prefer x_k to x_2. And by Pareto principle the society should prefer x_1 to x_k. Then, transitivity implies that the society prefers x_1 to x_2. The weak monotonicity implies that the society prefers x_1 to x_2 so long as individual i prefers x_1 to x_2, and individual i is decisive for x_1 against x_2. Therefore, individual i is the dictator, and we must assume that the society prefers x_1 to all other alternatives when the preference of individual i is $v_{(m-1)!+1}$ and the preferences of individuals other than i are v_1. □

In both AR and LP cases, by Pareto principle we obtain the correspondences from the vertices of S_i^1 to the vertices of S^1 by $f \circ \Delta$ as follows.

$$v_1 \sim v_{2(m-1)!} \longrightarrow w_2,\ v_{2(m-1)!+1} \sim v_{3(m-1)!} \longrightarrow w_3,\ v_{3(m-1)!+1} \sim v_{4(m-1)!} \longrightarrow w_1$$

[*5] From Lemma 1 of Baryshnikov (1993) we know that if individual preferences are strict orders, then the social preference is also a strict order under transitivity, Pareto principle and IIA.

All other vertices correspond to w_2. Sets of 1-dimensional simplices included in S_i^1 which are 1-dimensional cycles are only the following z and its counterpart $-z$.

$$z = < v_1, v_2 > + < v_2, v_3 > + \cdots + < v_{m!-1}, v_{m!} > + < v_{m!}, v_1 >$$

Since S_i^1 does not have a 2-dimensional simplex, z is a representative element of homology classes of S_i^1. z is transferred by $(f \circ \Delta)_*$ to the following z'.

$$z' =< w_2, w_3 > + < w_3, w_1 > + < w_1, w_2 >$$

This is a cycle of S^1. Therefore, we have

$$(f \circ \Delta)_* \neq 0 \tag{4.3}$$

Now we show the following lemma.

Lemma 4.3 1. If a binary social choice rule satisfies acyclicity, Pareto principle and the minimal liberalism described in Assumption 1, then we obtain

$$(f \circ i_i)_* = 0 \text{ for all } i \tag{4.4}$$

2. If a binary social choice rule satisfies transitivity, Pareto principle and IIA, then we obtain (4.4).

Proof. 1. First we show $(f \circ i_i)_* = 0$ for individual A and B. Consider the case of individual B. From Assumption 1 and Pareto principle, the correspondences from the preference of individual B to the social preference when the preference of every other individual (including individual A) is fixed at v_1 are obtained as follows.

$$v_1 \sim v_{(m-1)!} \longrightarrow w_2,\ v_{(m-1)!+1} \sim v_{m!} \longrightarrow w_1 \text{ or } w_2$$

In this case x_3 can not be the most preferred alternative for the society.
Sets of 1-dimensional simplices included in S_i^1 for individual B (denoted by S_B^1) which are 1-dimensional cycles are only the following z and its counterpart $-z$.

$$z = < v_1, v_2 > + < v_2, v_3 > + \cdots + < v_{m!-1}, v_{m!} > + < v_{m!}, v_1 >$$

Since S_B^1 does not have a 2-dimensional simplex, z is a representative element of homology classes of S_B^1. z is transferred by $(f \circ i_B)_*$, which is $(f \circ i_i)_*$ for individual B, to the following z' in S^1.

$$z' =< w_2, w_1 > + \cdots + < w_1, w_2 >= 0 \text{ , or } z' =< w_2, w_2 >= 0$$

This is not a cycle. Therefore, we get $(f \circ i_B)_* = 0$. Similarly we can show $(f \circ i_A)_* = 0$[*6].
Next we show $(f \circ i_i)_* = 0$ for any individual (denoted by i) other than A and B. From Assumption 1 and Pareto principle, the correspondences from the preference of individual i to the social preference when the preference of every other individual (including individual A and B) is fixed at v_1 are obtained as follows.

$$v_1 \sim v_{m!} \longrightarrow w_2$$

Because x_3 and x_4 can not be the most preferred alternative for the society. Then, we obtain $(f \circ i_i)_* = 0$ for all i other than A and B.

[*6] $(f \circ i_A)_*$ is $(f \circ i_i)_*$ for individual A. In this case x_4 can not be the most preferred alternative for the society.

2. By Pareto principle when the preference of every individual other than i is fixed at v_1, the correspondences from the preference of individual i to the social preference from v_1 to $v_{(m-1)!}$ are as follows.

$$v_1 \sim v_{(m-1)!} \longrightarrow w_2$$

From Lemma 4.2 the correspondence from $v_{(m-1)!+1}$ to the social preference is as follow.

$$v_{(m-1)!+1} \longrightarrow w_2$$

Consider another profile at which the preference of individual i changes to $(234 \cdots m1)$. By Pareto principle and the weak monotonicity (about x_1 and x_2) the society prefers x_1 to all other alternatives. Further the weak monotonicity implies that the society prefers x_1 to all other alternatives so long as the most preferred alternative for all individuals other than i is x_1 regardless of the preference of individual i. Thus, we obtain the following correspondences.

$$v_{(m-1)!+2} \sim v_{m!} \longrightarrow w_2$$

Sets of 1-dimensional simplices included in S_i^1 which are 1-dimensional cycles are only the following z and its counterpart $-z$.

$$z = < v_1, v_2 > + < v_2, v_3 > + \cdots + < v_{m!-1}, v_{m!} > + < v_{m!}, v_1 >$$

Since S_i^1 does not have a 2-dimensional simplex, z is a representative element of homology classes of S_i^1. z is transferred by $(f \circ i_i)_*$ to the following z'.

$$z' =< w_2, w_2 >= 0$$

Therefore, we have $(f \circ i_i)_* = 0$ for all i.

□

The conclusion of this lemma contradicts (4.2) and (4.3) for both LP and AR. Therefore, we have shown the following theorem.

Theorem 4.1 1. There exists no binary social choice rule which satisfies acyclicity, Pareto principle and the minimal liberalism.
2. There exists no binary social choice rule which satisfies transitivity, Pareto principle and IIA, and has no dictator.

We call the property expressed in (4.4) the *non-surjectivity of individual inclusion mappings*. Then, the above two theorems are special cases of the following theorem.

Theorem 4.2 There exists no binary social choice rule which satisfies Pareto principle and the non-surjectivity of individual inclusion mappings.

From (4.3) Pareto principle implies the *surjectivity of the diagonal mapping*, $(f \circ \Delta)_* \neq 0$, for binary social choice rules. Thus, this theorem is rewritten as follows.

> There exists no binary social choice rule which satisfies the *surjectivity of the diagonal mapping* and the *non-surjectivity of individual inclusion mappings*.

4.4 Concluding Remarks

We have shown the topological equivalence of the Arrow impossibility theorem that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives, and has no dictator, and Amartya Sen's liberal paradox that there exists no binary social choice rule which satisfies acyclicity, Pareto principle and the minimal liberalism. And we have also shown that these two theorems are special cases of the theorem that there exists no binary social choice rule which satisfies Pareto principle and the non-surjectivity of individual inclusion mappings.

In Baryshnikov (1997) he said, "the similarities between the two theories, the classical and topological ones, are somewhat more extended than one would expect. The details seem to fit too well to represent just an analogy. I would conjecture that the homological way of proving results in both theories is a 'true' one because of its uniformity and thus can lead to much deeper understanding of the structure of social choice. To understand this structure better we need a much more evolved collection of examples of unifying these two theories and I hope this can and will be done." This chapter is an attempt to provide such an example.

Chapter 5

A topological approach to Wilson's impossibility theorem

We will present a topological approach to Wilson's impossibility theorem (Wilson (1972)) that there exists no non-null binary social choice rule which satisfies transitivity, independence of irrelevant alternatives, non-imposition and has no dictator nor inverse dictator. Our research is in line with the studies of topological approaches to discrete social choice problems initiated by Baryshnikov (1993). This chapter extends the result about the Arrow impossibility theorem shown in Tanaka (2006b) to Wilson's theorem[*1]

5.1 Introduction

Topological approaches to social choice problems have been initiated by Chichilnisky (1980). Her main result is an impossibility theorem that there exists no *continuous* social choice rule which satisfies *unanimity* and *anonymity*. This approach has been further developed by Chichilnisky (1979), (1982), Koshevoy (1997), Lauwers (2004), Weinberger (2004) and so on. On the other hand, Baryshnikov (1993) and (1997) have presented a topological approach to the Arrow impossibility theorem (Arrow (1963)) in a discrete framework of social choice[*2]. In this chapter we will present a topological approach to Wilson's impossibility theorem (Wilson (1972)) that there exists no non-null binary social choice rule which satisfies transitivity, independence of irrelevant alternatives, non-imposition and has no dictator nor inverse dictator under the assumption of the *free triple property*. Our main tool is a homomorphism of homology groups of simplicial complexes induced by simplicial mappings[*3]. This chapter extends the result about the Arrow impossibility theorem shown in Tanaka (2006b) to Wilson's theorem. For other researches of topological approaches to social choice problems, see Tanaka (2006a), Tanaka (2006c) and Tanaka (2006d).

In the next section we summarize our model and preliminary results about the homology groups of simplicial complexes which represent individual and social preferences according to Tanaka (2006b). In Section 5.3 we will prove the main results.

[*1] This chapter is based on my paper of the same title published in *Journal of Mathematical Economics*, Vol. 43, No. 2, pp. 184-191, 2007, Elsevier.

[*2] About surveys and basic results of topological social choice theories, see Mehta (1997) and Lauwers (2000).

[*3] About homology groups we referred to Tamura (1970) and Komiya (2001).

5.2 The model and simplicial complexes

There are $n(\geq 3)$ alternatives and $k(\geq 2)$ individuals. n and k are finite positive integers. Denote individual i's preference by p_i. A profile of individual preferences is denoted by $\mathbf{p}$, and the set of profiles is denoted by $\mathcal{P}^k$. The alternatives are represented by x_i, $i = 1, 2, \cdots, n$. Individual preferences over the alternatives are weak orders, that is, individuals strictly prefer one alternative to another, or are indifferent between them. We consider a binary social choice rule which determines a social preference corresponding to a profile. Transitive social choice rule is called a *social welfare function* and is denoted by $F(\mathbf{p})$. We assume the free triple property, that is, for each combination of three alternatives individual preferences are not restricted. If the society is indifferent about every pair of two alternatives, the social welfare function is called *null*. If a social welfare function is not null, that is, the social preference derived by the social welfare function is strict about at least one pair of alternatives, then the social welfare function is called *non-null*.

Social welfare functions must be non-null, and must satisfy non-imposition and independence of irrelevant alternatives as well as transitivity. The meanings of the latter two conditions are as follows.

Non-imposition For every pair of two alternatives x_i and x_j there exists a profile at which the society prefers x_i to x_j or is indifferent between them.

Independence of irrelevant alternatives (IIA) The social preference about any pair of two alternatives x_i and x_j is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x_i and x_j.

The impossibility theorem by Wilson (1972) states that there exists no non-null binary social choice rule which satisfies transitivity, IIA, non-imposition and has no dictator nor inverse dictator. A dictator is an individual whose strict preference always coincides with the social preference, and an inverse dictator is an individual whose strict preference always coincides with the inverse of the social preference.

Hereafter we will consider a set of alternatives x_1, x_2 and x_3. From the set of individual preferences about x_1, x_2 and x_3 we construct a simplicial complex by the following procedures.

1. We denote a preference of an individual such that he prefers x_1 to x_2 by $(1, 2)$, a preference such that he prefers x_2 to x_1 by $(2, 1)$, a preference such that he is indifferent between x_1 and x_2 by $\overline{(1, 2)}$, and similar for other pairs of alternatives. Define vertices of the simplicial complex corresponding to (i, j) and $\overline{(i, j)}$.
2. A line segment between the vertices (i, j) and (k, l) is included in the simplicial complex if and only if the preference represented by (i, j) and the preference represented by (k, l) satisfy transitivity. For example, the line segment between $(1, 2)$ and $(3, 2)$ is included, but the line segment between $(1, 2)$ and $(2, 1)$ is not included in the simplicial complex.
3. A triangle (circumference plus interior) made by three vertices (i, j), (k, l) and (m, n) is included in the simplicial complex if and only if the preferences represented by (i, j), (k, l) and (m, n) satisfy transitivity. For example, a triangle made by $(1, 2)$, $(2, 3)$ and $(1, 3)$ is included in the simplicial complex. But a triangle made by $(1, 2)$, $(2, 3)$ and $(3, 1)$ is not included in the simplicial complex.

The simplicial complex constructed by these procedures is denoted by P. About a graphical presentation of the simplicial complexes see Tanaka (2006b).

We have shown the following result in Lemma 5.1 of Tanaka (2006b).

Lemma 5.1 The 1-dimensional homology group of P is isomorphic to the group of 6 integers, that is, $H_1(P) \cong \mathbb{Z}^6$.

Also about the simplicial complex, P^k, made by the set of profiles of individual preferences, $\mathcal{P}^k$, over x_1, x_2 and x_3 we have shown the following result in Lemma 5.2 of Tanaka (2006b).

Lemma 5.2 The 1-dimensional homology group of P^k is isomorphic to the group of $6k$ integers, that is, $H_1(P^k) \cong \mathbb{Z}^{6k}$.

The social preference about x_i and x_j is (i,j) or (j,i) or $\overline{(i,j)}$, and it is also represented by P. By the condition of IIA, individual preferences about alternatives other than x_i and x_j do not affect the social preference about them. Thus, the social welfare function F is a function from the vertices of P^k to the vertices of P. A set of points in P^k spans a simplex if and only if individual preferences represented by these points satisfy transitivity, and then the social preference derived from the profile represented by these points also satisfies transitivity. Therefore, if a set of points in P^k spans a simplex, the set of points in P which represent the social preference corresponding to those points in P^k also spans a simplex in P, and hence the social welfare function is a *simplicial mapping*. It is naturally extended from the vertices of P^k to all points in P^k. Each point in P^k is represented as a convex combination of the vertices of P^k. This function is also denoted by F.

We define an inclusion mapping from P to P^k, $\Delta : P \longrightarrow P^k : p \longrightarrow (p, p, \cdots, p)$, and an inclusion mapping which is derived by fixing the profile of preferences of individuals other than individual l to $\mathbf{p}_{-l}$, $i_l : P \longrightarrow P^k : p \longrightarrow (\mathbf{p}_{-l}, p)$. The homomorphisms of 1-dimensional homology groups induced by these inclusion mappings are

$$\Delta_* : \mathbb{Z}^6 \longrightarrow \mathbb{Z}^{6k} : h \longrightarrow (h, h, \cdots, h),\ h \in \mathbb{Z}^6$$

$$i_{l*} : \mathbb{Z}^6 \longrightarrow \mathbb{Z}^{6k} : h \longrightarrow (0, \cdots, h, \cdots, 0) \text{ (only the } l\text{-th component is } h \text{ and others are zero, } h \in \mathbb{Z}^6)$$

From these definitions about Δ_* and i_{l*} we obtain the following relation.

$$\Delta_* = i_{1*} + i_{2*} + \cdots + i_{k*} \tag{5.1}$$

And the homomorphism of homology groups induced by F is represented as follows.

$$F_* : \mathbb{Z}^{6k} \longrightarrow \mathbb{Z}^6 : \mathbf{h} = (h_1, h_2, \cdots, h_k) \longrightarrow h,\ h \in \mathbb{Z}^6$$

The composite function of i_l and the social welfare function F is $F \circ i_l : P \longrightarrow P$, and its induced homomorphism satisfies $(F \circ i_l)_* = F_* \circ i_{l*}$. The composite function of Δ and F is $F \circ \Delta : P \longrightarrow P$, and its induced homomorphism satisfies $(F \circ \Delta)_* = F_* \circ \Delta_*$. From (5.1) we have

$$(F \circ \Delta)_* = (F \circ i_1)_* + (F \circ i_2)_* + \cdots + (F \circ i_k)_*$$

$F \circ i_l$ when the profile of individuals other than individual l is $\mathbf{p}_{-l}$ and $F \circ i_l$ when the profile of individuals other than individual l is $\mathbf{p}'_{-l}$ are homotopic. Thus, the induced homomorphism $(F \circ i_l)_*$ of $F \circ i_l$ does not depend on the preferences of individuals other than l.

For a pair of alternatives x_i and x_j, a profile, at which all individuals prefer x_i to x_j, is denoted by $(i, j)^{(+)}$; a profile, at which they prefer x_j to x_i, is denoted by $(i, j)^{(-)}$. And a profile, at which the preferences of all individuals about x_i and x_j are not specified, is denoted by $(i, j)^s$ where $s = \{+, 0, -\}^k$ with s_j the sign of individual j. 0 denotes indifference. Similarly a profile, at which all individuals other than l prefer x_i to x_j, is denoted by $(i, j)^{(+)}_{-l}$; a profile, at which they prefer x_j to x_i, is denoted by $(i, j)^{(-)}_{-l}$. And a profile, at which the preferences of individuals other than l about x_i and x_j are not specified, is denoted by $(i, j)^s_{-l}$.

5.3 The main results

First we show the following lemma.

Lemma 5.3 If $(F \circ \Delta)_* = 0$ the society is indifferent about any pair of alternatives, that is, the social welfare function is null.

Proof. Consider a set of three alternatives, x_1, x_2 and x_3. Assume that when all individuals prefer x_1 to x_2, the society prefers x_1 to x_2 (or prefers x_2 to x_1), that is, assume the following correspondence from individual preferences to the social preference.

$$(1, 2)^{(+)} \longrightarrow (1, 2) \text{ [or } (2, 1)]$$

By non-imposition there exists a profile such that we have the following correspondences.

$$(2, 3)^s \longrightarrow (2, 3) \text{ or } (\overline{2, 3}) \text{ [or ``}(3, 2) \text{ or } (\overline{2, 3})\text{'']}$$
$$(1, 3)^s \longrightarrow (3, 1) \text{ or } (\overline{1, 3}) \text{ [or ``}(1, 3) \text{ or } (\overline{1, 3})\text{'']}$$

Transitivity implies

$$(1, 3)^{(+)} \longrightarrow (1, 3) \text{ [or } (3, 1)], \tag{5.2}$$
$$(2, 3)^{(-)} \longrightarrow (3, 2) \text{ [or } (2, 3)] \tag{5.3}$$

Again, by non-imposition there exists a profile such that we have the correspondence.

$$(1, 2)^s \longrightarrow (2, 1) \text{ or } (\overline{1, 2}) \text{ [or ``}(1, 2) \text{ or } (\overline{1, 2})\text{'']}$$

Then, from transitivity we obtain

$$(1, 3)^{(-)} \longrightarrow (3, 1) \text{ [or } (1, 3)],$$
$$(2, 3)^{(+)} \longrightarrow (2, 3) \text{ [or } (3, 2)]$$

From these arguments we find that a cycle of P, $z =< (1, 2), (2, 3) > + < (2, 3), (3, 1) > - < (1, 2), (3, 1) >$, corresponds to a cycle $z =< (1, 2), (2, 3) > + < (2, 3), (3, 1) > - < (1, 2), (3, 1) >$, or a cycle $z' =< (2, 1), (3, 2) > + < (3, 2), (1, 3) > - < (2, 1), (1, 3) >$ of P for the social preference by $(F \circ \Delta)_*$. Because both z and z' are not a boundary cycle, we have $(F \circ \Delta)_* \neq 0$. This result can be reached starting from an assumption other than $(1, 2)^{(+)} \longrightarrow (1, 2)$ [or $(1, 2)^{(+)} \longrightarrow (2, 1)$], for example, $(2, 3)^{(+)} \longrightarrow (2, 3)$ [or $(2, 3)^{(+)} \longrightarrow (3, 2)$].

Therefore, if $(F \circ \Delta)_* = 0$ we obtain the following correspondences from individual preferences to the social preference.

$$\left.\begin{array}{ll} (1, 2)^{(+)} \longrightarrow (\overline{1, 2}), & (2, 3)^{(+)} \longrightarrow (\overline{2, 3}) \\ (2, 3)^{(-)} \longrightarrow (\overline{2, 3}), & (1, 3)^{(+)} \longrightarrow (\overline{1, 3}) \end{array}\right\} \tag{5.4}$$

From (5.4) with transitivity we obtain

$$(1,3)^s \longrightarrow \overline{(1,3)},\quad (2,3)^s \longrightarrow \overline{(2,3)},\quad (1,2)^s \longrightarrow \overline{(1,2)}$$

Thus, the society is indifferent about any pair of alternatives among x_1, x_2 and x_3.

Interchanging x_3 with x_4 in the proof of this lemma, we can show that the society is indifferent about any pair of alternatives among x_1, x_2 and x_4. Similarly, the society is indifferent among x_5, x_2 and x_4, and it is indifferent among x_5, x_6 and x_4. After all the society is indifferent about any pair of alternatives, that is, the social welfare function is null. □

This lemma implies that if a social welfare function is non-null, we have $(F \circ \Delta)_* \neq 0$. Further we show the following lemma.

Lemma 5.4 1. If individual l is a dictator or an inverse dictator, we have $(F \circ i_l)_* \neq 0$.
2. If he is not a dictator nor inverse dictator, we have $(F \circ i_l)_* = 0$.

Proof. 1. Consider three alternatives x_1, x_2 and x_3. If individual l is a dictator, the correspondences from his preference to the social preference by $F \circ i_l$ are as follows,

$$(1,2)_l \longrightarrow (1,2),\quad (2,1)_l \longrightarrow (2,1),\quad (2,3)_l \longrightarrow (2,3),$$

$$(3,2)_l \longrightarrow (3,2),\quad (1,3)_l \longrightarrow (1,3),\quad (3,1)_l \longrightarrow (3,1)$$

$(1,2)_l$ and $(2,1)_l$ denote the preference of individual l about x_1 and x_2. $(2,3)_l$, $(3,2)_l$ and so on are similar. These correspondences imply that a cycle of P, $z = <(1,2),(2,3)> + <(2,3),(3,1)> - <(1,2),(3,1)>$, corresponds to the same cycle of P for the social preference by $(F \circ i_l)_*$. Because z is not a boundary cycle, we have $(F \circ i_l)_* \neq 0$.

On the other hand, if individual l is an inverse dictator, the correspondences from his preference to the social preference by $F \circ i_l$ are as follows,

$$(1,2)_l \longrightarrow (2,1),\quad (2,1)_l \longrightarrow (1,2),\quad (2,3)_l \longrightarrow (3,2)$$

$$(3,2)_l \longrightarrow (2,3),\quad (1,3)_l \longrightarrow (3,1),\quad (3,1)_l \longrightarrow (1,3)$$

These correspondences imply that a cycle of P, $z = <(1,2),(2,3)> + <(2,3),(3,1)> - <(1,2),(3,1)>$, corresponds to a cycle $z' = <(2,1),(3,2)> + <(3,2),(1,3)> - <(2,1),(1,3)>$ of P for the social preference by $(F \circ i_l)_*$, and so we have $(F \circ i_l)_* \neq 0$.

2. From the proof of Lemma 5.3 if a social welfare function is non-null, there are the following two cases.

(a) Case (a): The following four correspondences simultaneously hold.

$$\left.\begin{array}{l} (1,2)^{(+)} \longrightarrow (1,2),\quad (1,3)^{(+)} \longrightarrow (1,3) \\ (2,3)^{(+)} \longrightarrow (2,3),\quad (2,3)^{(-)} \longrightarrow (3,2) \end{array}\right\} \tag{5.5}$$

(b) Case (b): The following four correspondences simultaneously hold.

$$\left.\begin{array}{l} (1,2)^{(+)} \longrightarrow (2,1),\quad (1,3)^{(+)} \longrightarrow (3,1) \\ (2,3)^{(+)} \longrightarrow (3,2),\quad (2,3)^{(-)} \longrightarrow (2,3) \end{array}\right\} \tag{5.6}$$

We will provide the proof of Case (b). The proof of Case (a) is similar.

Consider three alternatives x_1, x_2 and x_3 and a profile $\mathbf{p}$ over them such that the preferences of individuals other than l are represented by $(1,2)^{(+)}_{-l}$, $(2,3)^{(+)}_{-l}$ and $(1,3)^{(+)}_{-l}$. If individual l is not an inverse dictator, there exists a profile at which the social preference about some pair of alternatives does not coincide with the inverse of his strict preference. Assume that when the preference of individual l is $(1,2)$, the social preference is $(1,2)$ or $(\overline{2,1})$. Then, we obtain the following correspondence from the profile to the social preference.

$$(1,2)^s_{-l} \times (1,2)_l \longrightarrow (1,2) \text{ or } (\overline{1,2})$$

Then, from (5.6) and transitivity we obtain

$$(2,3)^{(+)}_{-l} \times (3,2)_l \longrightarrow (3,2)$$

and

$$(1,3)^{(+)}_{-l} \times (3,1)_l \longrightarrow (3,1)$$

Further, from (5.6) and transitivity we get the following correspondence.

$$(1,2)^{(+)}_{-l} \times (2,1)_l \longrightarrow (2,1)$$

These results imply that at a profile **p**, where the preferences of individuals other than l are represented by $(1,2)^{(+)}_{-l}$, $(2,3)^{(+)}_{-l}$ and $(1,3)^{(+)}_{-l}$, the correspondences from the preference of individual l to the social preference by $F \circ i_l$ are as follows.

$$(1,2)_l \longrightarrow (2,1),\ (2,1)_l \longrightarrow (2,1),\ (2,3)_l \longrightarrow (3,2),$$
$$(3,2)_l \longrightarrow (3,2),\ (1,3)_l \longrightarrow (3,1),\ (3,1)_l \longrightarrow (3,1)$$

These correspondences with transitivity and IIA imply that, when individual l is indifferent between x_1 and x_3, the society prefers x_3 to x_1, that is, we obtain the following correspondence.

$$(\overline{1,3})_l \longrightarrow (3,1)$$

This is derived from two correspondences $(1,2)_l \longrightarrow (2,1)$ and $(3,2)_l \longrightarrow (3,2)$. Therefore, the following four sets of correspondences are impossible because the correspondences in each set are not consistent with the correspondence $(\overline{1,3})_l \longrightarrow (3,1)$.

(a) $(\overline{1,2})_l \longrightarrow (\overline{1,2}),\ (\overline{2,3})_l \longrightarrow (\overline{2,3})$
(b) $(\overline{1,2})_l \longrightarrow (\overline{1,2}),\ (\overline{2,3})_l \longrightarrow (2,3)$
(c) $(\overline{1,2})_l \longrightarrow (1,2),\ (\overline{2,3})_l \longrightarrow (2,3)$
(d) $(\overline{1,2})_l \longrightarrow (1,2),\ (\overline{2,3})_l \longrightarrow (\overline{2,3})$

There are the following five cases, which are consistent with the correspondence $(\overline{1,3})_l \longrightarrow (3,1)$.

(a) Case (i): $(\overline{1,2})_l \longrightarrow (\overline{1,2}),\ (\overline{2,3})_l \longrightarrow (3,2)$
(b) Case (ii): $(\overline{1,2})_l \longrightarrow (2,1),\ (\overline{2,3})_l \longrightarrow (\overline{2,3})$
(c) Case (iii): $(\overline{1,2})_l \longrightarrow (2,1),\ (\overline{2,3})_l \longrightarrow (3,2)$
(d) Case (iv): $(\overline{1,2})_l \longrightarrow (1,2),\ (\overline{2,3})_l \longrightarrow (3,2)$
(e) Case (v): $(\overline{1,2})_l \longrightarrow (2,1),\ (\overline{2,3})_l \longrightarrow (2,3)$

We consider Case (i). The arguments for other cases are similar.

In Case (i) we have $(\overline{1,2})_l \longrightarrow (\overline{1,2}),\ (\overline{2,3})_l \longrightarrow (3,2)$. The vertices of P for the social preference mapped from the preference of individual l by $F \circ i_l$ span the following five simplices.

$$<(2,1),(3,2)>,\ <(2,1),(3,1)>,\ <(3,2),(3,1)>,\ <(\overline{1,2}),(3,2)>,$$
$$<(\overline{1,2}),(3,1)>$$

Then, an element of the 1-dimensional chain group is written as

$$c_1 = a_1 < (2,1),(3,2) > + a_2 < (2,1),(3,1) > + a_3 < (3,2),(3,1) > \\ + a_4 < (\overline{1,2}),(3,2) > + a_5 < (\overline{1,2}),(3,1) >,\ a_i \in \mathbb{Z}$$

The condition for an element of the 1-dimensional chain group to be a cycle is

$$\partial c_1 = (-a_1 - a_2) < (2,1) > + (a_1 - a_3 + a_4) < (3,2) > \\ + (a_2 + a_3 + a_5) < (3,1) > + (-a_4 - a_5) < (\overline{1,2}) > = 0$$

From this

$$-a_1 - a_2 = 0,\ a_1 - a_3 + a_4 = 0,\ a_2 + a_3 + a_5 = 0,\ -a_4 - a_5 = 0$$

are derived. Then, we obtain $a_2 = -a_1, a_5 = -a_4, a_3 = a_1 + a_4$. Therefore, an element of the 1-dimensional cycle group, Z_1, is written as follows.

$$z_1 = a_1 < (2,1),(3,2) > - a_1 < (2,1),(3,1) > + (a_1 + a_4) < (3,2),(3,1) > \\ + a_4 < (\overline{1,2}),(3,2) > - a_4 < (\overline{1,2}),(3,1) >$$

On the other hand, the vertices span the following 2-dimensional simplices.

$$< (2,1),(3,2),(3,1) >,\ < (\overline{1,2}),(3,2),(3,1) >$$

Then, an element of the 2-dimensional chain group is written as

$$c_2 = b_1 < (2,1),(3,2),(3,1) > + b_2 < (\overline{1,2}),(3,2),(3,1) >,\ b_i \in \mathbb{Z}$$

And an element of the 1-dimensional boundary cycle group, B_1, is written as follows.

$$\partial c_2 = b_1 < (2,1),(3,2) > - b_1 < (2,1),(3,1) > + (b_1 + b_2) < (3,2),(3,1) > \\ + b_2 < (\overline{1,2}),(3,2) > - b_2 < (\overline{1,2}),(3,1) >$$

Then, we find that B_1 is isomorphic to Z_1, and so the 1-dimensional homology group is trivial, that is, we have proved $(F \circ i_l)_* = 0$.
Thus, if there exists no inverse dictator, we have $(F \circ i_l)_* = 0$.

□

From these arguments and $(F \circ \Delta)_* \neq 0$ there exists a dictator or an inverse dictator about x_1, x_2 and x_3. Let individual l be a dictator or an inverse dictator. Interchanging x_3 with x_4 in the proof of this lemma, we can show that he is a dictator or an inverse dictator about x_1, x_2 and x_4. Similarly, we can show that he is a dictator or an inverse dictator about x_5, x_2 and x_4, he is a dictator or an inverse dictator about x_5, x_6 and x_4. After all he is a dictator or an inverse dictator about all alternatives

From these lemmas we obtain the following theorem.

Theorem 5.1 (Wilson's impossibility theorem) There exists a dictator or an inverse dictator for a social welfare function which is non-null, and satisfies IIA and non-imposition.

Proof. From Lemma 5.3 if a social welfare function is non-null, we have $(F \circ \Delta)_* \neq 0$. Therefore, from Lemma 5.4 there exists a dictator or an inverse dictator. □

Chapter 6

Equivalence of the HEX game theorem and the Arrow impossibility theorem

Gale (1979) has shown that the so called HEX game theorem that any HEX game has one winner is equivalent to the Brouwer fixed point theorem. In this chapter we will show that under some assumptions about marking rules of HEX games, the HEX game theorem is equivalent to the Arrow impossibility theorem of social choice theory that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator. We assume that individual preferences over alternatives are strong (or linear) orders, that is, the individuals are not indifferent about any pair of alternatives[*1].

6.1 Introduction

Gale (1979) has shown that the so called HEX game theorem that any HEX game has one winner is equivalent to the Brouwer fixed point theorem. In this chapter we will show that under some assumptions about marking rules of HEX games, the HEX game theorem is equivalent to the Arrow impossibility theorem of social choice theory (Arrow (1963)) that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator. We assume that individual preferences over alternatives are strong (or linear) orders, that is, the individuals are not indifferent about any pair of alternatives.

In the next section according to Gale (1979) we present an outline of the HEX game. In Section 6.3 we will show that the HEX game theorem implies the Arrow impossibility theorem. And in Section 6.4 we will show that the Arrow impossibility theorem implies the HEX game theorem.

6.2 The HEX game

According to Gale (1979) we present an outline of the HEX game. Figure 1 (a) represents a 6×6 HEX board. Generally a HEX game is represented by an $n \times n$ HEX board where n is a finite positive integer. The rules of the game are as follows. Two players (called Mr. W and Mr. B) move alternately, marking any previously unmarked hexagon or tile with a white (by Mr. W) or a black (by Mr. B) circle respectively.

[*1] This chapter is based on my paper of the same title published in *Applied Mathematics and Computation*, Vol. 186, No. 1, pp. 509-515, 2007, Elsevier.

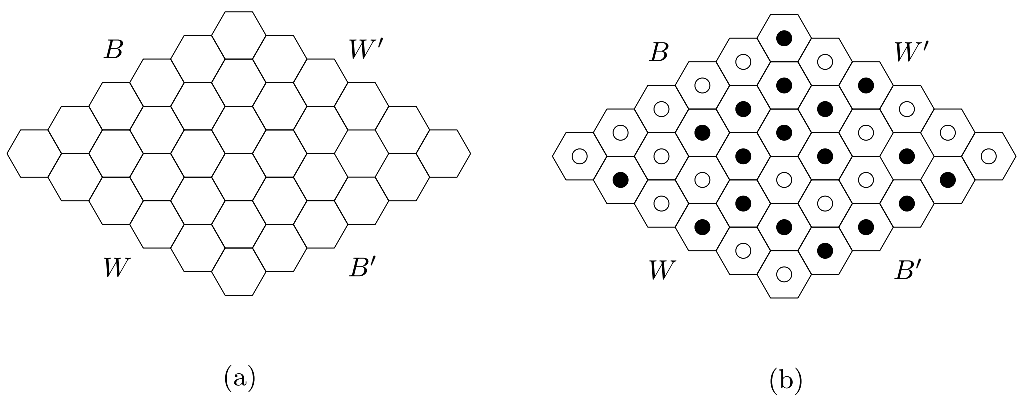


Figure 1: HEX game

The game has been won by Mr. W (or Mr. B) if he has succeeded in marking a connected set of tiles which meets the boundary regions W and W' (or, B and B'). A set S of tiles is connected if any two members of the set h and h' can be joined by a path $P = (h = h^1, h^2, \dots, h^m = h')$ where h^i and h^{i+1} are adjacent. Figure 1 (b) represents a HEX game which has been won by Mr. B.

About the HEX game Gale (1979) has shown the following theorem.

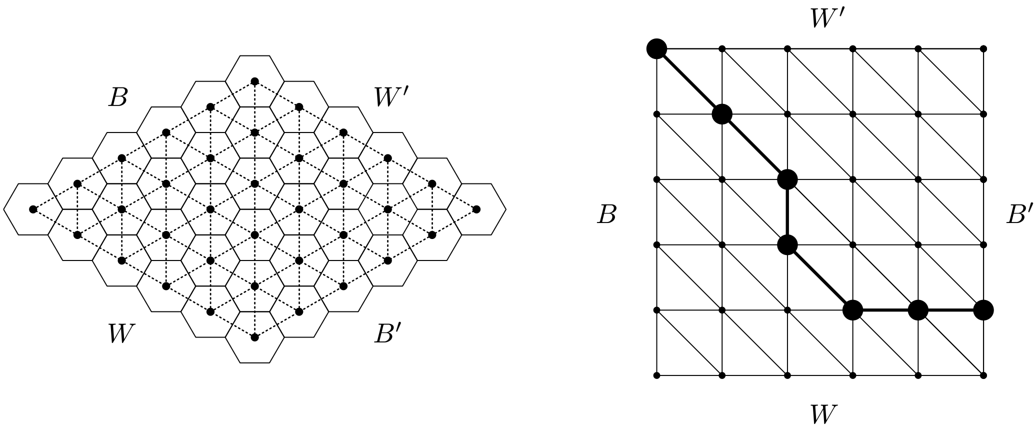
Theorem 6.1 (The HEX game theorem) If every tile of the HEX board is marked by either a white or a black circle, then there is a path connecting regions W and W', or a path connecting regions B and B'.

Actually he has shown the theorem that any hex game can never end in a draw, and there always exists at least one winner. But, from his intuitive explanation using the following example of river and dam, it is clear that there exists only one winner of any hex game.

> Imagine that B and B' regions are portions of opposite banks of the river which flow from W region to W' region, and that Mr. B is trying to build a dam by putting down stones. He will have succeeded in damming the river if and only if he has placed his stones in a way which enables him to walk on them from one bank (B region) to the other (B' region).

The proof of Theorem 6.1 and also the above intuitive argument do not depend on the rule "two players move *alternately*". Therefore, this theorem is valid for any marking rule.

Figure 2 (a) is obtained by plotting the center of each hexagon, and connecting these centers by lines. Rotating this graph 45° in anticlockwise direction, we obtain Figure 2 (b). It is an equivalent representation of the HEX board depicted in Figure 1 (a). W and W' represent the regions of Mr. W, and B and B' represent the regions of Mr. B. We call it a square HEX board, and call a game represented by a square HEX board a *square HEX game*. In Figure 2 (b) we depict an example of winning marking by Mr. B. It corresponds to the marking pattern in Figure 1 (b). A set of marked vertices which represents one player's victory is called a *winning path*.



(a) (b)

Figure 2: Square HEX game and winning path

6.3 The HEX game theorem implies the Arrow impossibility theorem

There are $m(\geq 3)$ alternatives and $n(\geq 2)$ individuals. m and n are finite positive integers. The set of individuals is denoted by N. Denote individual i's preference by p_i. A combination of individual preferences, which is called a *profile*, is denoted by $\mathbf{p}(= (p_1, p_2, \cdots, p_n))$. The set of profiles is denoted by $\mathcal{P}^n$. The alternatives are represented by $x_i,\ i = 1, 2, \cdots, m$. Individual preferences over the alternatives are strong (or linear) orders, that is, individuals strictly prefer one alternative to another, and are not indifferent about any pair of alternatives. We assume the free triple property, that is, for each set of three alternatives individual preferences are never restricted.

We consider a binary social choice rule which determines a social preference corresponding to each profile. Binary social choice rules must satisfy the conditions of *transitivity*, *Pareto principle* and *independence of irrelevant alternatives (IIA)*. Transitive binary social choice rules are called *social welfare functions*. The meanings of these conditions are as follows.

Transitivity If, according to a social welfare function, the society prefers an alternative x_i to another alternative x_j, and prefers x_j to another alternative x_k, then the society must prefer x_i to x_k.

Pareto principle When all individuals prefer x_i to x_j, the society must prefer x_i to x_j.

Independence of irrelevant alternatives (IIA) The social preference about every pair of two alternatives x_i and x_j is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x_i and x_j.

From Lemma 1 of Baryshnikov (1993) we know that if individual preferences are strong orders, then the social preference is also a strong order under the conditions of transitivity, Pareto principle and IIA.

The Arrow impossibility theorem states that there exists no social welfare function which has no dictator, or in other words there exists a dictator for any social welfare function. A dictator is an individual whose strict preference always coincides with the social preference.

According to Sen (1979) we define the following terms.

Almost decisiveness If, when all individuals in a group G prefer an alternative x_i to another alternative x_j, and the other individuals (individuals in $N \setminus G$) prefer x_j to x_i, the society prefers x_i to x_j, then G is *almost decisive* for x_i against x_j.

Decisiveness If, when all individuals in a group G prefer an alternative x_i to another alternative x_j, the society prefers x_i to x_j regardless of the preferences of the other individuals, then G is *decisive* for x_i against x_j.

G may consists of one individual. By Pareto principle N is almost decisive and decisive about every pair of alternatives. If for a social welfare function an individual is decisive about every pair of alternatives, then he is the *dictator* of the social welfare function.

Sen (1979) and Suzumura (2000) have shown the following result.

Lemma 6.1 (Lemma 3*a in Sen (1979) and Dictator Lemma in Suzumura (2000)) If one individual is almost decisive for one alternative against another alternative, then he is the dictator of the social welfare function.

This lemma holds under the conditions of transitivity, Pareto principle and IIA. The conclusion of this lemma is also valid in the case where not an individual but a group of individuals is almost decisive for one alternative against another alternative. Thus, the following lemma is derived.

Lemma 6.2 If a group of individuals G is almost decisive for one alternative against another alternative, then this group is decisive about every pair of alternatives.

Now we confine us to a subset of profiles $\bar{\mathcal{P}}^n$ such that all individuals prefer three alternatives x_1, x_2 and x_3 to all other alternatives. Pareto principle implies that at all such profiles the society also prefers x_1, x_2 and x_3 to all other alternatives. We denote individual preferences about x_1, x_2 and x_3 in this subset of profiles as follows.

$$p^1 = (123),\ p^2 = (132),\ p^3 = (312),\ p^4 = (321),\ p^5 = (231),\ p^6 = (213)$$

$p^1 = (123)$ represents all preferences such that an individual prefers x_1 to x_2 to x_3 to all other alternatives, and so on. Although we confine our arguments to such a subset of profiles, Lemma 6.1 with IIA ensures that an individual who is almost decisive about a pair of alternatives for this subset of profiles is the dictator for all profiles.

From Lemma 6.2 for the profiles in $\bar{\mathcal{P}}^n$ we obtain the following result.

Lemma 6.3 If two groups G and G', which are not disjoint, are almost decisive about a pair of alternatives, then their intersection $G \cap G'$ is decisive about every pair of alternatives.

Proof. By Lemma 6.2 G and G' are decisive about every pair of alternatives. For three alternatives x_1, x_2 and x_3 we consider the following profile in $\bar{\mathcal{P}}^n$.

1. Individuals in $G \setminus (G \cap G')$ prefer x_3 to x_1 to x_2.
2. Individuals in $G' \setminus (G \cap G')$ prefer x_2 to x_3 to x_1.
3. Individuals in $G \cap G'$ prefer x_1 to x_2 to x_3.
4. Individuals in $N \setminus (G \cup G')$ prefer x_3 to x_2 to x_1.

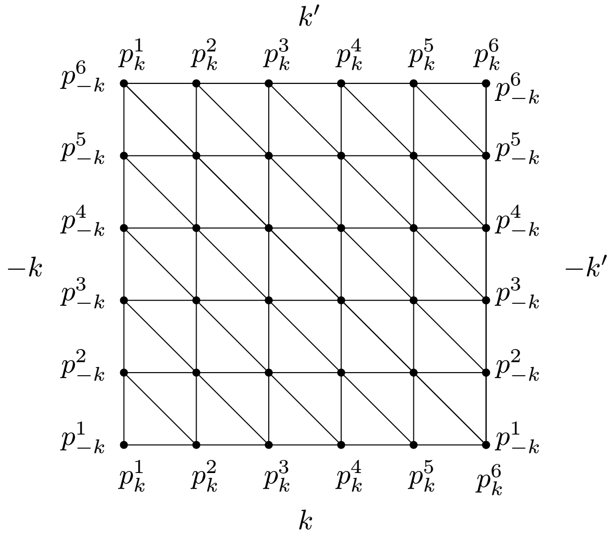


Figure 3: HEX game representing profiles

By the decisiveness of G and G' and transitivity the society must prefer x_1 to x_2 to x_3. Since only individuals in $G \cap G'$ prefer x_1 to x_3 and all other individuals prefer x_3 to x_1, $G \cap G'$ is almost decisive for x_1 against x_3 under IIA. From Lemma 6.2 it is decisive about every pair of alternatives. □

Further we confine us to a subset of $\bar{\mathcal{P}}^n$ such that all but one individual have the same preferences, and consider a HEX game between one individual (denoted by individual k) and the set of individuals other than k. Representative profiles are denoted by (p_k^i, p_{-k}^j), $i = 1, \ldots, 6$, $j = 1, \ldots, 6$, where p_k^i is individual k's preference and p_{-k}^j denotes the common preference of the individuals other than k. We relate these profiles to the vertices of a 6×6 square HEX board as depicted in Figure 3. There are 36 vertices in this HEX board. It represents a square HEX game. k and k' represent individual k's regions, and $-k$ and $-k'$ represent the regions of the set of individuals other than k.

We consider the following marking and winning rules of the square HEX game.

1. At a profile represented by a vertex of a square HEX board, if the society's most preferred alternative is the same as that of individual k and different from that of the individuals other than k, then this vertex is marked by a white circle; conversely if the society's most preferred alternative is the same as that of the individuals other then k and different from that of individual k, then this vertex is marked by a black circle.
   Hereafter we abbreviate the most preferred alternative by MPA.
2. At a profile, if the society's MPA is different from any of those of individual k and the individuals other than k, or the MPAs of all individuals are the same, then the vertex which corresponds to this profile is randomly marked by a white or a black circle.
3. The game has been won by individual k (or the set of individuals other than k) if he has (or they have) succeeded in marking a connected set of vertices which meets the boundary regions k and k' (or $-k$ and $-k'$).

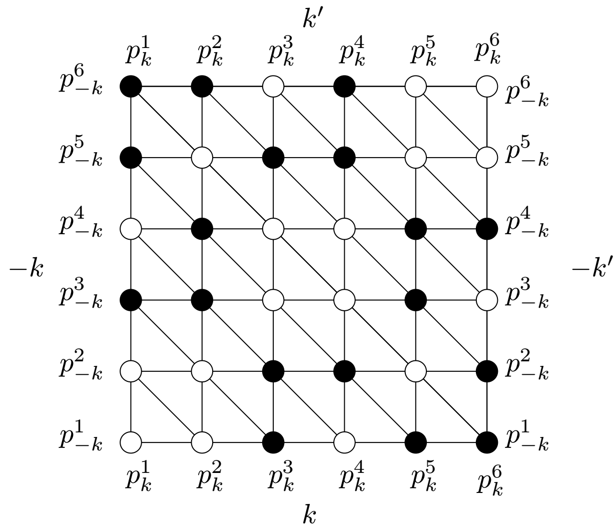


Figure 4: Winning path of a square HEX game

A square HEX game is equivalent to the original HEX game. Therefore, there exists one winner for any marking rule. Now we show the following result.

Theorem 6.2 The HEX game theorem implies the existence of a dictator for any social welfare function.

Proof. Since diagonal vertices are not connected, the diagonal path

$$((p_k^1, p_{-k}^1), (p_k^2, p_{-k}^2), \cdots, (p_k^6, p_{-k}^6))$$

can not be a winning path. Consider a profile $(p_k^1, p_{-k}^6) = ((123), (213))$. By Pareto principle x_3 is not the society's MPA. Suppose that at this profile the society's MPA is x_2 which is the MPA of the individuals other than k. Then, by Pareto principle and IIA the society's MPA at the following profiles is x_2.

$$(p_k^1, p_{-k}^5),\ (p_k^2, p_{-k}^6)$$

The fact that the society's MPA at a profile (p_k^2, p_{-k}^6) is x_2, with Pareto principle and IIA, implies that the society's MPA at the following profiles is x_2.

$$(p_k^3, p_{-k}^5),\ (p_k^4, p_{-k}^5),\ (p_k^4, p_{-k}^6)$$

Similarly consider a profile $(p_k^3, p_{-k}^2) = ((312), (132))$. By Pareto principle x_2 is not the society's MPA. Suppose that at this profile the society's MPA is x_1 which is the MPA of the individuals other than k. Then, by Pareto principle and IIA the society's MPA at the following profiles is x_1.

$$(p_k^3, p_{-k}^1),\ (p_k^4, p_{-k}^2)$$

The fact that the society's MPA at a profile (p_k^4, p_{-k}^2) is x_1, with Pareto principle and IIA, implies that the society's MPA at the following profiles is x_1.

$$(p_k^5, p_{-k}^1),\ (p_k^6, p_{-k}^1),\ (p_k^6, p_{-k}^2)$$

Similarly consider a profile $(p_k^5, p_{-k}^4) = ((231), (321))$. By Pareto principle x_1 is not the society's MPA. Suppose that at this profile the society's MPA is x_3 which is the MPA of the individuals other than k. Then, by Pareto principle and IIA the society's MPA at the following profiles is x_3.

$$(p_k^5, p_{-k}^3),\ (p_k^6, p_{-k}^4)$$

The fact that the society's MPA at a profile (p_k^6, p_{-k}^4) is x_3, with Pareto principle and IIA, implies that the society's MPA at the following profiles is x_3.

$$(p_k^1, p_{-k}^3),\ (p_k^2, p_{-k}^3),\ (p_k^2, p_{-k}^4)$$

The vertices which correspond to all of these profiles are marked by black circles. Then, when all other vertices are marked by white circles, we obtain a marking pattern of a square HEX board as depicted in Figure 4. The set of individuals other than k is the winner of this HEX game. Therefore, for individual k to be the winner of a square HEX game, the society's MPA must coincide with that of individual k at least at one of three profiles (p_k^1, p_{-k}^6), (p_k^3, p_{-k}^2) and (p_k^5, p_{-k}^4). It means that individual k must be almost decisive about at least one pair of alternatives, and then by Lemma 6.1 he is the dictator.

If for all k, $(k = 1, 2, \cdots, n)$, individual k is not the winner of any square HEX game between individual k and the set of individuals other than k, then each set of individuals excluding one individual is the winner of each square HEX game. By Lemma 6.3 every nonempty intersection of the sets of individuals excluding one individual is decisive. Then, the intersection of $N \setminus \{1\}$, $N \setminus \{2\}$, $\cdots$, $N \setminus \{n-1\}$ is decisive. But $(N \setminus \{1\}) \cap (N \setminus \{2\}) \cap \cdots (N \setminus \{n-1\}) = \{n\}$. Thus, individual n is the dictator. Therefore, the HEX game theorem implies the existence of a dictator for any social welfare function. □

By this theorem the HEX game theorem implies the Arrow impossibility theorem.

6.4 The Arrow impossibility theorem implies the HEX game theorem

Next we will show that the Arrow impossibility theorem implies the HEX game theorem under an interpretation of dictator. Similarly to the previous section, we confine us to a subset of profiles such that all individuals prefer three alternatives x_1, x_2 and x_3 to all other alternatives, and the preferences of individuals other than one individual (denoted by k) are the same. And we consider a square HEX game between individual k and the set of individuals other than k. The dictator of a social welfare function is interpreted as an individual who can determine the MPA of the society when his MPA and that of the other individuals are different, and in a HEX game he can mark tiles with his color in such cases. We denote a vertex of a square HEX board which corresponds to a profile (p_k^i, p_{-k}^j) simply by (p_k^i, p_{-k}^j).

First, consider the case where individual k is the dictator of a social welfare function. Then, the following vertices are marked by white circles.

$$(p_k^1, p_{-k}^3),\ (p_k^1, p_{-k}^4),\ (p_k^1, p_{-k}^5),\ (p_k^1, p_{-k}^6),\ (p_k^2, p_{-k}^3),\ (p_k^2, p_{-k}^4),\ (p_k^2, p_{-k}^5),\ (p_k^2, p_{-k}^6)$$

$$(p_k^3, p_{-k}^1),\ (p_k^3, p_{-k}^2),\ (p_k^3, p_{-k}^5),\ (p_k^3, p_{-k}^6),\ (p_k^4, p_{-k}^1),\ (p_k^4, p_{-k}^2),\ (p_k^4, p_{-k}^5),\ (p_k^4, p_{-k}^6)$$

$$(p_k^5, p_{-k}^1),\ (p_k^5, p_{-k}^2),\ (p_k^5, p_{-k}^3),\ (p_k^5, p_{-k}^4),\ (p_k^6, p_{-k}^1),\ (p_k^6, p_{-k}^2),\ (p_k^6, p_{-k}^3),\ (p_k^6, p_{-k}^4)$$

We obtain Figure 5. Unmarked vertices, where the MPAs of all individuals are the same, should be randomly marked. Clearly individual k is the winner of this HEX game.

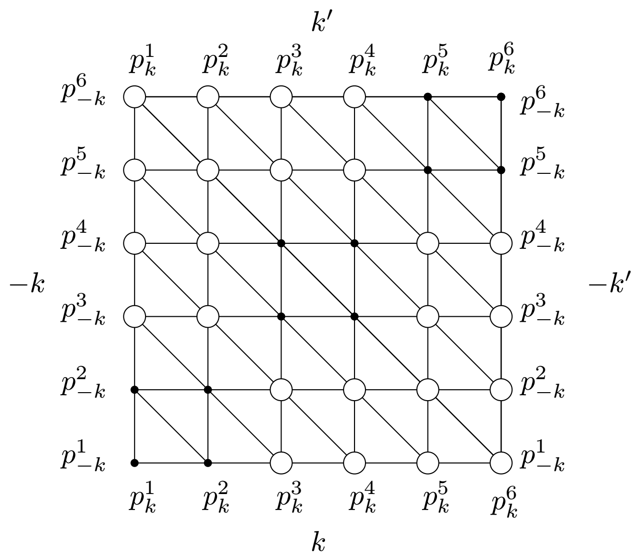


Figure 5: HEX game won by individual k

Next, consider the case where the dictator of a social welfare function is included in the set of individuals other than k. Then, all of the above vertices are marked by black circles, and the set of individuals other than k is the winner of the HEX game.

Therefore, the existence of a dictator for a social welfare function implies the existence of a winner for a HEX game, and we obtain

Theorem 6.3 The Arrow impossibility theorem and the HEX game theorem are equivalent.

6.5 Concluding Remarks

We have considered the relationship between the HEX game theorem and the Arrow impossibility theorem, and have shown their equivalence. In this chapter we have assumed that individual preferences over alternatives are strong (or linear) orders. We are now proceeding research on extension of the result of this chapter to the case where individual preferences over alternatives are weak orders, that is, they may be indifferent about any pair of two alternatives (See Chapter 8).

Chapter 7

On the equivalence of the HEX game theorem and the Duggan-Schwartz theorem for strategy-proof social choice correspondences

Gale (1979) has shown that the so called HEX game theorem that any HEX game has one winner is equivalent to the Brouwer fixed point theorem. In this chapter we will show that under some assumptions about marking rules of HEX games, the HEX game theorem for a 6×6 HEX game is equivalent to the Duggan-Schwartz theorem for strategy-proof social choice correspondences (Duggan and Schwartz (2000)) that there exists no social choice correspondence which satisfies the conditions of strategy-proofness, non-imposition, residual resoluteness, and has no dictator[*1].

7.1 Introduction

Gale (1979) has shown that the so called HEX game theorem that any HEX game has one winner is equivalent to the Brouwer fixed point theorem. In this chapter we will show that under some assumptions about marking rules of HEX games, the HEX game theorem for a 6×6 HEX game is equivalent to the Duggan-Schwartz theorem for strategy-proof social choice correspondences (Duggan and Schwartz (2000)) that there exists no social choice correspondence which satisfies the conditions of strategy-proofness, non-imposition, residual resoluteness, and has no dictator[*2].

In the next section according to Gale (1979) we present an outline of the HEX game. In Section 7.3 we will show that the HEX game theorem implies the Duggan-Schwartz theorem. And in Section 7.4 we will show that the Duggan-Schwartz theorem implies the HEX game theorem.

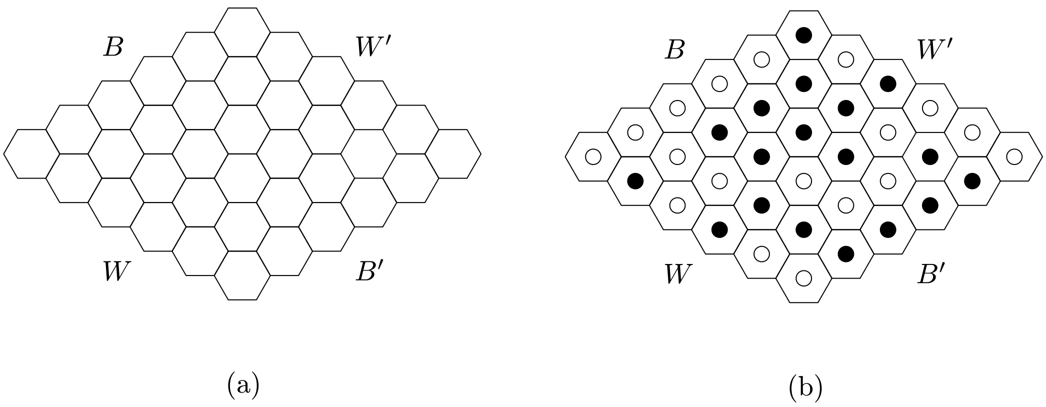


Figure 1: HEX game

7.2 The HEX game

According to Gale (1979) we present an outline of the HEX game. Figure 1 (a) represents a 6×6 HEX board[*3]. Generally a HEX game is represented by an $n \times n$ HEX board where n is a finite positive integer. The rules of the game are as follows. Two players (called Mr. W and Mr. B) move alternately, marking any previously unmarked hexagon or tile with a white (by Mr. W) or a black (by Mr. B) circle respectively. The game has been won by Mr. W (or Mr. B) if he has succeeded in marking a connected set of tiles which meets the boundary regions W and W' (or, B and B'). A set S of tiles is connected if any two members of the set h and h' can be joined by a path $P = (h = h^1, h^2, \dots, h^m = h')$ where h^i and h^{i+1} are adjacent. Figure 1 (b) represents a HEX game which has been won by Mr. B.

About the HEX game Gale (1979) has shown the following theorem.

Theorem 7.1 (The HEX game theorem) If every tile of the HEX board is marked by either a white or a black circle, then there is a path connecting regions W and W', or a path connecting regions B and B'.

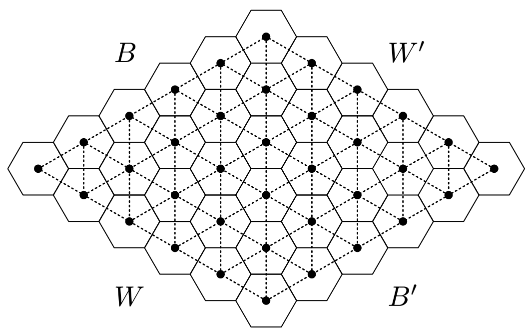
Actually he has shown the theorem that any hex game can never end in a draw, and there always exists at least one winner. But, from his intuitive explanation using the following example of river and dam, it is clear that there exists only one winner in any hex game.

> Imagine that B and B' regions are portions of opposite banks of the river which flow from W region to W' region, and that Mr. B is trying to build a dam by putting down stones. He will have succeeded in damming the river if and only if he has placed his stones in a way which enables him

[*1] This chapter is based on my paper of the same title published in *Applied Mathematics and Computation*, Vol. 188, No. 1, pp. 303-313, 2007, Elsevier.

[*2] In another paper, Tanaka (2007a), we have shown the equivalence of the HEX game theorem and the Arrow impossibility theorem. This chapter will apply this idea to the problem of the existence of a dictator for strategy-proof social choice correspondences.

[*3] About the HEX game see also Binmore (1991).



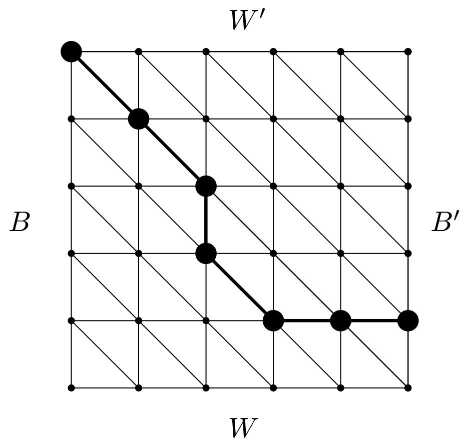


Figure 2: Square HEX game and winning path

to walk on them from one bank (B region) to the other (B' region).

The proof of Theorem 7.1 and also the above intuitive argument do not depend on the rule "two players move *alternately*". Therefore, this theorem is valid for any marking rule.

Figure 2 (a) is obtained by plotting the center of each hexagon, and connecting these centers by lines. Rotating this graph 45° in anticlockwise direction, we obtain Figure 2 (b). It is an equivalent representation of the HEX board depicted in Figure 1 (a). W and W' represent the regions of Mr. W, and B and B' represent the regions of Mr. B. We call it a square HEX board, and call a game represented by a square HEX board a *square HEX game*. In Figure 2 (b) we depict an example of winning marking by Mr. B. It corresponds to the marking pattern in Figure 1 (b). A set of marked vertices which represents one player's victory is called a *winning path*.

7.3 The HEX game theorem implies the Duggan-Schwartz theorem

There are $m(\geq 3)$ alternatives and $n(\geq 2)$ individuals. m and n are finite positive integers. The set of individuals is denoted by N, the set of alternatives is denoted by A, and the set of all subsets of A is denoted by $\mathcal{A}$. Denote individual i's preference by P_i. A combination of individual preferences, which is called a *profile*, is denoted by $p(= (P_1, P_2, \cdots, P_n))$, and the set of profiles is denoted by $\mathcal{P}^n$. The alternatives are represented by x, y, z and so on. Individual preferences over the alternatives are strong (or linear) orders, that is, individuals strictly prefer one alternative to another, and are not indifferent about any pair of alternatives.

We consider a social choice correspondence which chooses one or more alternatives corresponding to each profile of *revealed preferences* of the individuals. It is a mapping of $\mathcal{P}^n$ into $\mathcal{A}$. Given profiles p, p', p'' ... we denote by $C(p)$, $C(p')$, $C(p'')$ ... the set of alternatives chosen by a social choice correspondence at each profile. We call $C(p)$ the *social choice set* at p, $C(p')$ the social choice set at p', and so on. $C(p)$ for any p is not empty.

We assume the conditions of non-imposition (or citizens' sovereignty) and residual resoluteness. The means of these conditions are as follows.

Non-imposition For any social choice correspondence and every alternative x there is a profile p at which x is chosen by a social choice correspondence, that is, $x \in C(p)$.

Residual resoluteness Assume that at a profile p all but one (denoted by i) individual have the same preferences, and they most prefer an alternative x and secondly prefer another alternative y. And assume that individual i has the same preference as those of other individuals, or only x and y are interchanged in his preference. Then, $C(p)$ is a singleton (the social choice correspondence chooses only one alternative).

As demonstrated by Duggan and Schwartz (2000) residual resoluteness is an appropriate condition if the number of individuals is not so small.

Next we consider strategy-proofness according to the definition by Duggan and Schwartz (2000). We assume that each individual (represented by i) has a von Neumann-Morgenstern utility function u_i. If $u_i(x) > u_i(y)$ when individual i prefers x to y for arbitrary pair of alternatives (x, y), the preference of individual i *is represented by* u_i.

Let p and p' be two profiles between which only the preference of individual i differs. $C(p)$ and $C(p')$ are the social choice sets at p and p'. Assume that individual i assigns probabilities $\mu(x)$ and $\mu'(x)$ to an alternative x included in $C(p)$ and $C(p')$, and so on. $\mu(x)$ is individual i's subjective probability of alternative x when $C(p)$ is the social choice set, and $\mu'(x)$ has similar meaning. Then, his expected utilities at p and p' evaluated by his utility function at p are

$$E_i(p) = \sum_{x \in C(p)} \mu(x) u_i(x) \text{ (where } \sum_{x \in C(p)} \mu(x) = 1)$$

and

$$E_i(p') = \sum_{x \in C(p')} \mu'(x) u_i(x) \text{ (where } \sum_{x \in C(p')} \mu'(x) = 1)$$

If for *all* assignments of probabilities to alternatives we have

$$E_i(p') > E_i(p), \tag{7.1}$$

then individual i has an incentive to report his preference P'_i when his true preference is P_i, and the social choice correspondence is *manipulable* by him at p. Conversely, if for *some* assignment of probabilities we have $E_i(p) \geqq E_i(p')$, then the social choice correspondence is not manipulable.

Now we can show the following lemma.

Lemma 7.1 Let p and p' be two profiles of individual preferences between which only the preference of one individual (denoted by i) differs. If and only if for some $x \in C(p')$ and all $y \in C(p)$, or for some $y \in C(p)$ and all $x \in C(p')$ individual i prefers x to y at p, the social choice correspondence is manipulable by him at p.

Proof. First consider the case where for some $x \in C(p')$ and all $y \in C(p)$ individual i prefers x to y at p. Let $\varepsilon > 0$ be the probability of x assigned by him at p', w be his top-ranked (most preferred) alternative in $C(p)$, v be his bottom-ranked (least preferred) alternative in $C(p')$ evaluated by his utility function at p, u_i. Then, we obtain

$$E_i(p') \geqq \varepsilon u_i(x) + (1-\varepsilon)u_i(v)$$

and

$$E_i(p) \leqq u_i(w)$$

Since $u_i(x) > u_i(w)$ and $u_i(x) \geqq u_i(v)$, given ε we can determine the value of $u_i(x)$ so that $E_i(p') > E_i(p)$ holds.

Next consider the case where for some $y \in C(p)$ and all $x \in C(p')$ individual i prefers x to y at p. Let $\varepsilon > 0$ be the probability of y assigned by him at p, w be his bottom-ranked (least preferred) alternative in $C(p')$, and v be his top-ranked (most preferred) alternative in $C(p)$ evaluated by his utility function at p, u_i. Then we obtain

$$E_i(p') \geqq u_i(w)$$

and

$$E_i(p) \leqq \varepsilon u_i(y) + (1-\varepsilon)u_i(v)$$

Since $u_i(y) < u_i(w)$ and $u_i(y) \leqq u_i(v)$, given ε we can determine the value of $u_i(y)$ so that $E_i(p') > E_i(p)$ holds.

Finally, assume that there exists no $x \in C(p')$ such that individual i prefers x to y for all $y \in C(p)$, and no $y \in C(p)$ such that he prefers x to y for all $x \in C(p')$ at p. Let x be his top-ranked alternative in $C(p')$ and y be his bottom-ranked alternative in $C(p)$ evaluated by his utility function at p, u_i. Then, there exists at least one $w \in C(p)$ such that individual i prefers w to x and at least one $z \in C(p')$ such that individual i prefers y to z at p. Let ε' and ε be, respectively, the probability of z at p' and the probability of w at p assigned by him. Then, we obtain

$$E_i(p') \leqq \varepsilon' u_i(z) + (1-\varepsilon')u_i(x)$$

and

$$E_i(p) \geqq \varepsilon u_i(w) + (1-\varepsilon)u_i(y)$$

Since $u_i(w) > u_i(x)$ and $u_i(y) > u_i(z)$, if we assume $\varepsilon = 1 - \varepsilon'$, we obtain $E_i(p) > E_i(p')$, and (7.1) does not hold. □

Taylor (2002) and Taylor (2005) defined that a social choice correspondence is *manipulated by an optimist* in the case where for some $x \in C(p')$ and all $y \in C(p)$ individual i prefers x to y at p, and defined that it is *manipulated by an pessimist* in the case where for some $y \in C(p)$ and all $x \in C(p')$ individual i prefers x to y at p.

Strategy-proofness is defined as follows:

Strategy-proofness If a social choice correspondence is not manipulable for any individual at any profile, it is *strategy-proof*.

Further we define some terminologies as follows.

Monotonicity Let $C(p)$ be the social choice set at some profile p, y be an alternative outside of $C(p)$, and assume the following individual preferences at p.

1. Individuals in a group S $(S \subseteq N)$ prefer x to y for some $x \in C(p)$.

2. Others (group $S' = N - S$) prefer y to x for all $x \in C(p)$.

Consider another profile $p' \in \mathcal{P}^n$ such that individuals in S are partitioned into the following l sub-groups*4.

1. S_1: For some set of alternatives X_1 which includes $C(p)$ and does not include y ($C(p) \subset X_1$ and $y \notin X_1$), individuals in S_1 prefer x' to z for all $x' \in X_1$ and all $z \notin X_1$ at p'.
2. S_2: For some set of alternatives X_2 which includes X_1 and does not include y ($X_1 \subset X_2$ and $y \notin X_2$), individuals in S_2 prefer x' to z for all $x' \in X_2$ and all $z \notin X_2$ at p'.
3. $\cdots\cdots$
4. S_{l-1}: For some set of alternatives X_{l-1} which includes X_{l-2} and does not include y ($X_{l-2} \subset X_{l-1}$ and $y \notin X_{l-1}$), individuals in S_{l-1} prefer x' to z for all $x' \in X_{l-1}$ and all $z \notin X_{l-1}$ at p',
5. S_l: Their preferences do not change.

The preferences of individuals in S' at p' are not specified. Then, the social choice correspondence does not choose y at p' ($y \notin C(p')$).

Semi-decisive A group of individuals S is *semi-decisive* for x against y if when for some set of alternatives X such that $x \in X$ and $y \notin X$ individuals in S prefer x' to z for all $x' \in X$ and all $z \notin X$, a social choice correspondence does not choose y regardless of the preferences of other individuals.

Semi-decisive set If S is semi-decisive about all pairs of alternatives, it is called a semi-decisive set.

S in the definition of semi-decisive set may consists of one individual. If, for a social choice correspondence, a set of one individual is a semi-decisive set, then this individual is a dictator of the social choice correspondence because any alternative other than his most preferred alternative is never chosen.

Now we show the following result.

Lemma 7.2 If a social choice correspondence is strategy-proof, then it satisfies monotonicity.

In the following proof we use notations in the definition of monotonicity, and we neglect individuals in S_l whose preferences do not change between p and p'.

Proof. Without loss of generality let individuals 1 to m ($0 \leqq m \leqq n$) belong to S and individuals $m+1$ to n belong to S'. Consider a profile p'' other than p and p' such that individuals in S prefer x to y to z for all $x \in C(p)$, and individuals in S' prefer y to x to z for all $x \in C(p)$, where z is an arbitrary alternative other than alternatives in $C(p)$ and y.

Let p^1 be a profile such that only the preference of individual 1 changes from P_1 (his preference at p) to P_1'' (his preference at p''), and suppose that at p^1 an alternative other than alternatives in $C(p)$ is included in the social choice set. Then, he has an incentive to report a false preference P_1 when his true preference is P_1'' because he prefers alternatives in $C(p)$ to all other alternatives at p^1. Therefore, at p^1 only alternatives in $C(p)$ are chosen by the social choice correspondence. By the same logic, when the preferences of individuals 1 to m change from P_i to P_i'', only alternatives in $C(p)$ are chosen. Next, let p^{m+1} be a profile such that the preference of individual $m+1$, as well as the preferences of the first m individuals, changes from P_{m+1} to P_{m+1}'', and suppose that at p^{m+1} y is included in the social choice set. Then, individual $m+1$ has an incentive to report a false preference P_{m+1}'' when his true preference is P_{m+1} because at p he prefers y to x for all $x \in C(p)$. On the other hand, if an alternative other than alternatives in $C(p)$ is included in the social choice set at p^{m+1}, he has an incentive to report a false

*4 The number of sub-groups l does not exceed the number of individuals who belong to S.

preference P_{m+1} when his true preference is P''_{m+1} because at p^{m+1} he prefers x to z for all $x \in C(p)$ and all $z \notin C(p)$, $z \neq y$. By the same logic, when the preferences of all individuals change from P_i to P''_i, only alternatives in $C(p)$ are chosen by the social choice correspondence.

Now, suppose that from p'' to p' the individual preferences change one by one from P''_i to P'_i. If, when the preference of the first individual in S_1 changes, an alternative outside of X_1 is chosen, he has an incentive to report a false preference P''_i when his true preference is P'_i because at p' he prefers x to z for all $x \in X_1$ and all $z \notin X_1$. Consequently only some alternatives included in X_1 are chosen. By the same logic, when the preferences of all individuals in S_1 change from P''_i to P'_i, only some alternatives in X_1 are chosen. Similarly, when the preferences of all individuals in S_2 (denoted by j) change from P''_j to P'_j, only some alternatives in X_2 are chosen, $\cdots$, when the preferences of all individuals in S_{l-1} (denoted by k) change from P''_k to P'_k, only some alternatives in X_{l-1} are chosen. Further, if, when the preference of the first individual (individual $m+1$) in S' changes, y is included in the social choice set, then he has an incentive to report a false preference P'_{m+1} when his true preference is P''_{m+1} because at p'' he prefers y to z for all $z \neq y$. By the same logic, when the preferences of all individuals change, y is not chosen by the social choice correspondence. □

The Duggan-Schwartz theorem states that there exists a dictator for any strategy-proof social choice correspondence which satisfies the conditions of non-imposition and residual resoluteness, or in other words, there exists no social choice correspondence which satisfies the conditions of strategy-proofness, non-imposition, residual resoluteness, and has no dictator. A dictator for a social choice correspondence is an individual such that the social choice correspondence always chooses only his most preferred alternative, or in other words the social choice set always includes only his most preferred alternative.

About the concepts of semi-decisiveness and semi-decisive set we will show some results. As preliminary results we show the following lemmas.

Lemma 7.3 (Unanimity) Suppose that a social choice correspondence satisfies the conditions of strategy-proofness, non-imposition and residual resoluteness. If at a profile p all individuals most prefer an alternative (denoted by x), then the social choice correspondence chooses only this alternative, that is, $C(p) = \{x\}$.

Proof. Consider a profile p' at which all individuals have the same preferences and they most prefer x. By residual resoluteness only one alternative is chosen by the social choice correspondence. By non-imposition at some profile p'' x is chosen ($x \in C(p'')$). If, when the preference of one individual (individual 1) changes from P''_1 (his preference at p'') to P'_1 (his preference at p'), x is not chosen by the social choice correspondence, then individual 1 has an incentive to report a false preference P''_1 when his true preference is P'_1 because he most prefers x at p', and the social choice correspondence is manipulable by individual 1. Thus, x is chosen in this case. By the same logic x is chosen at p'. By residual resoluteness at p' only x is chosen ($C(p') = \{x\}$).

Next, if, when the preference of one individual (individual 1) changes from P'_1 to P_1 (his preference at p), an alternative other than x is chosen by the social choice correspondence, then he has an incentive to report a false preference P'_1 when his true preference is P_1 because he most prefers x at p, and the social choice correspondence is manipulable by individual 1. By the same logic any alternative other than x is not chosen at p, and we have $C(p) = \{x\}$. □

Lemma 7.4 Suppose that a social choice correspondence satisfies the conditions of strategy-proofness, non-imposition and residual resoluteness.

1. Let partition the individuals into the following two groups, and for alternatives x, y and w we assume the following profile p:
   (a) individuals in a group S: $xP_i yP_i wP_i z$
   (b) others: $yP_i wP_i xP_i z$
   where z denotes an arbitrary alternative other than x, y and w. Then, the social choice correspondence does not choose any alternative other than x and y.
2. Similarly, let partition the individuals into the following two groups, and for alternatives x, y and w we assume the following profile p:
   (a) individuals in S: $wP_i xP_i yP_i z$
   (b) others (N/S): $yP_i wP_i xP_i z$
   where z denotes an arbitrary alternative other than x, y and w. Then, the social choice correspondence does not choose any alternative other than y and w.

Proof. 1. By Lemma 7.3 there is a profile p' at which $C(p') = \{y\}$. Suppose that, starting from individuals outside of S, their preferences change from P_i' to P_i (from profile p' to p) one by one. Even when the preferences of individuals outside of S change, only y is chosen because they most prefer y at p. On the other hand, when the preferences of individuals in S change, any alternative other than x and y is not chosen because they most prefer x and secondly prefer y at p.
2. Permuting w, x and y and interchanging S and N/S, the proof of this case is the same as the proof of Case 1.

□

Next we show

Lemma 7.5 Suppose that a social choice correspondence satisfies the conditions of strategy-proofness, non-imposition and residual resoluteness. If a group S is semi-decisive about one pair of alternatives, then it is a semi-decisive set.

Proof. Assume that S is semi-decisive for x against y. Let w be an alternative other than x and y.

1. Consider the following profile p.
   (a) Individuals in S prefer x to y to w to z.
   (b) Other individuals prefer y to w to x to z.
   z denotes an arbitrary alternative other than x, y and w. Since S is semi-decisive for x against y we have $y \notin C(p)$. From Lemma 7.4 we have $w \notin C(p)$ and $z \notin C(p)$, and so we have $C(p) = \{x\}$. Individuals in S prefer x to w, but all other individuals prefer w to x. Therefore, by monotonicity S is semi-decisive for x against w.
2. Next consider the following profile p'.
   (a) Individuals in S prefer w to x to y to z.
   (b) Other individuals prefer y to w to x to z.
   z denotes an arbitrary alternative other than x, y and w. Since S is semi-decisive for x against y we have $y \notin C(p)$. From Lemma 7.4 we have $x \notin C(p)$ and $z \notin C(p)$, and so we have $C(p) = \{w\}$.

Individuals in S prefer w to y, but all other individuals prefer y to w. Therefore, by monotonicity S is semi-decisive for w against y.

Applying this logic repeatedly we can show that S is a semi-decisive set. □

The implications of this lemma are similar to those of Lemma 3*a in Sen (1979) and Dictator Lemma in Suzumura (2000) for binary social choice rules. If a set of one individual is semi-decisive about one pair of alternatives, he is a dictator.

Now we confine us to a subset of profiles $\bar{\mathcal{P}}^n$ such that all individuals prefer three alternatives x, y and z to all other alternatives. Unanimity implies that the set of all individuals N is semi-decisive about every pair of alternatives, and so it is a semi-decisive set. Thus, by monotonicity any social choice correspondence does not choose any alternative other than x, y and z at all such profiles. We denote individual preferences about x, y and z in this subset of profiles as follows.

$$p^1 = (123),\ p^2 = (132),\ p^3 = (312),\ p^4 = (321),\ p^5 = (231),\ p^6 = (213)$$

$p^1 = (123)$ represents all preferences such that an individual prefers x to y to z to all other alternatives, $p^1 = (132)$ represents all preferences such that an individual prefers x to z to y to all other alternatives, and so on. Although we confine our arguments to such a subset of profiles, Lemma 7.5 with monotonicity ensures that an individual who is semi-decisive about a pair of alternatives for this subset of profiles is a dictator for all profiles.

From Lemma 7.5 for the profiles in $\bar{\mathcal{P}}^n$ we obtain the following result.

Lemma 7.6 If two groups S and S', which are not disjoint, are semi-decisive sets, then their intersection $S \cap S'$ is a semi-decisive set.

Proof. For three alternatives x, y and z we consider the following profile.

1. Individuals in $S \setminus (S \cap S')$ prefer z to x to y.
2. Individuals in $S' \setminus (S \cap S')$ prefer y to z to x.
3. Individuals in $S \cap S'$ prefer x to y to z.
4. Individuals in $N \setminus (S \cup S')$ prefer z to y to x.

Since S and S' are semi-decisive sets, the social choice correspondence does not choose y and z. Thus, it chooses x. Since only individuals in $S \cap S'$ prefer x to z and all other individuals prefer z to x, by monotonicity $S \cap S'$ is semi-decisive for x against z. From Lemma 7.5 it is a semi-decisive set. □

Further we confine us to a subset of $\bar{\mathcal{P}}^n$ such that all but one individual have the same preferences, and consider a HEX game between one individual (denoted by individual k) and the set of individuals other than k. Representative profiles are denoted by (p^i_k, p^j_{-k}), $i = 1, \dots, 6$, $j = 1, \dots, 6$, where p^i_k is individual k's preference and p^j_{-k} denotes the common preference of individuals other than k. We relate these profiles to the vertices of a 6×6 square HEX board as depicted in Figure 3. There are 36 vertices in this HEX board. It represents a square HEX game. k and k' represent individual k's regions, and $-k$ and $-k'$ represent the regions of the set of individuals other than k.

We consider the following marking and winning rules of square HEX games.

1. At a profile represented by a vertex of a square HEX board, if the social choice correspondence chooses *only* the most preferred alternative of individual k which is different from the most preferred

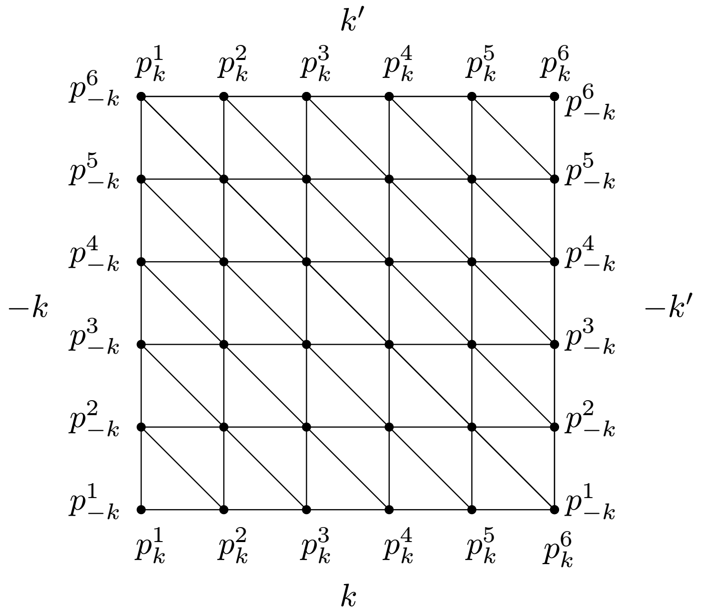


Figure 3: HEX game representing profiles

alternative of the individuals other than k, then this vertex is marked by a white circle; conversely if the social choice correspondence chooses only the most preferred alternative of the individuals other then k which is different from the most preferred alternative of individual k, then this vertex is marked by a black circle.

2. A vertex which corresponds to any other profile is randomly marked by a white or a black circle.
3. The game has been won by individual k (or the set of individuals other than k) if he has (or they have) succeeded in marking a connected set of vertices which meets the boundary regions k and k' (or $-k$ and $-k'$).

A square HEX game is equivalent to the original HEX game. Therefore, there exists one winner for any marking rule. Now we show the following theorem

Theorem 7.2 The HEX game theorem implies the existence of a dictator for any social choice correspondence which satisfies the conditions of strategy-proofness, non-imposition and residual resoluteness.

Proof. Since diagonal vertices are not connected, the diagonal path

$$((p_k^1, p_{-k}^1), (p_k^2, p_{-k}^2), \cdots, (p_k^6, p_{-k}^6))$$

can not be a winning path. Consider a profile $(p_k^1, p_{-k}^6) = ((123), (213))$. Unanimity (Lemma 7.3) and monotonicity mean that z is not chosen by the social choice correspondence at this profile*5.

By residual resoluteness only one alternative is chosen. Suppose that at this profile the social choice correspondence chooses only y which is the most preferred alternative of the individuals other than k.

*5 Here X_1 in the definition of monotonicity is $\{x, y\}$.

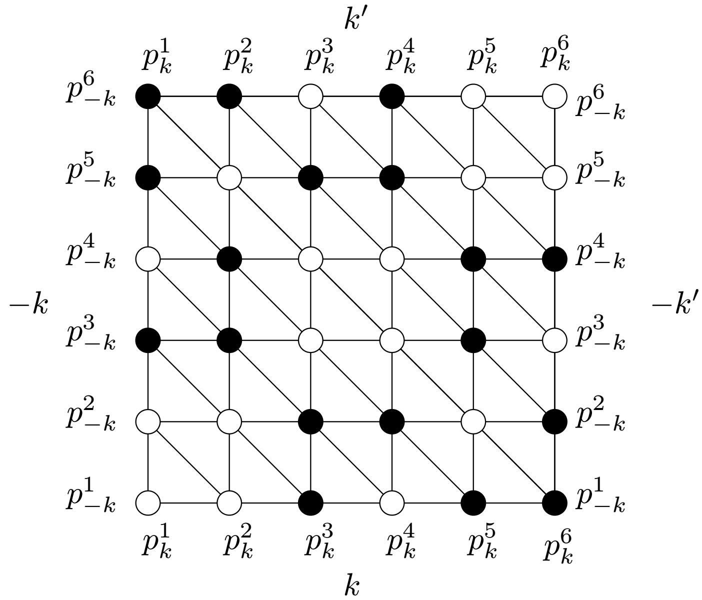


Figure 4: Winning path of a square HEX game

Then, by monotonicity the social choice correspondence chooses only y at the following profiles.

$$(p_k^1, p_{-k}^5),\ (p_k^2, p_{-k}^6)$$

[Note] At a profile (p_k^1, p_{-k}^1) only x is chosen by unanimity. Then, monotonicity means that z is not chosen at (p_k^1, p_{-k}^6) and (p_k^2, p_{-k}^6)[*6]. The fact that at (p_k^1, p_{-k}^6) only y is chosen and monotonicity imply that z is not chosen at (p_k^1, p_{-k}^5), and imply that x is not chosen at (p_k^1, p_{-k}^5) and (p_k^2, p_{-k}^6). We can apply similar arguments to other cases.

The fact that the social choice correspondence chooses only y at a profile (p_k^2, p_{-k}^6) and monotonicity imply that the social choice correspondence chooses only y at the following profiles.

$$(p_k^3, p_{-k}^5),\ (p_k^4, p_{-k}^5),\ (p_k^4, p_{-k}^6)$$

Similarly consider a profile $(p_k^3, p_{-k}^2) = ((312), (132))$. By unanimity and monotonicity y is not chosen by the social choice correspondence. By residual resoluteness only one alternative is chosen. Suppose that at this profile the social choice correspondence chooses only x which is the most preferred alternative of the individuals other than k. Then, by monotonicity the social choice correspondence chooses only x at the following profiles.

$$(p_k^3, p_{-k}^1),\ (p_k^4, p_{-k}^2)$$

The fact that the social choice correspondence chooses only x at a profile (p_k^4, p_{-k}^2) and monotonicity imply that the social choice correspondence chooses only x at the following profiles.

$$(p_k^5, p_{-k}^1),\ (p_k^6, p_{-k}^1),\ (p_k^6, p_{-k}^2)$$

*6 Here X_1 and X_2 in the definition of monotonicity are $\{x\}$ and $\{x, y\}$.

Similarly consider a profile $(p_k^5, p_{-k}^4) = ((231), (321))$. By unanimity and monotonicity x is not chosen by the social choice correspondence. By residual resoluteness only one alternative is chosen. Suppose that at this profile the social choice correspondence chooses only z which is the most preferred alternative of the individuals other than k. Then, by monotonicity the social choice correspondence chooses only z at the following profiles.

$$(p_k^5, p_{-k}^3),\ (p_k^6, p_{-k}^4)$$

The fact that the social choice correspondence chooses only z at a profile (p_k^6, p_{-k}^4) and monotonicity imply that the social choice correspondence chooses only z at the following profiles.

$$(p_k^1, p_{-k}^3),\ (p_k^2, p_{-k}^3),\ (p_k^2, p_{-k}^4)$$

The vertices which correspond to all of these profiles are marked by black circles. Then, even when all other vertices are marked by white circles, we obtain a marking pattern of a square HEX board as depicted in Figure 4. The set of individuals other than k is the winner of this game. Therefore, for individual k to be the winner of the square HEX game, the alternative chosen by the social choice correspondence must coincide with the most preferred alternative of individual k at least at one of three profiles (p_k^1, p_{-k}^6), (p_k^3, p_{-k}^2) and (p_k^5, p_{-k}^4). Then, by monotonicity individual k is semi-decisive about at least one pair of alternatives, and then by Lemma 7.5 he is a dictator.

If for all k, $(k = 1, 2, \cdots, n)$, individual k is not the winner of all square HEX games between individual k and the set of individuals other than k, then each set of individuals excluding one individual is the winner of each square HEX game. By Lemma 7.6 every nonempty intersection of the sets of individuals excluding one individual is a semi-decisive set. Then, the intersection of $N \setminus \{1\}$, $N \setminus \{2\}$, $\cdots$, $N \setminus \{n-1\}$ is a semi-decisive set. But $(N \setminus \{1\}) \cap (N \setminus \{2\}) \cap \cdots (N \setminus \{n-1\}) = \{n\}$. Thus, individual n is a dictator. Therefore, the HEX game theorem implies the existence of a dictator for any social choice correspondence which satisfies the conditions of strategy-proofness, non-imposition and residual resoluteness. □

By this theorem the HEX game theorem implies the Duggan-Schwartz theorem.

7.4 The Duggan-Schwartz theorem implies the HEX game theorem

Next we show that the Duggan-Schwartz theorem implies the HEX game theorem under an interpretation of dictator. Similarly to the previous section, we confine us to a subset of profiles such that all individuals prefer three alternatives x, y and z to all other alternatives, and the preferences of individuals other than one individual (denoted by k) are the same. And we consider a square HEX game between individual k and the set of individuals other than k. The dictator of a social choice correspondence is interpreted as an individual who can determine the choice of the society when his most preferred alternative and that of the other individuals are different, and in a HEX game he can mark tiles with his color in such cases. Without loss of generality we assume that individual k is a dictator of a social choice correspondence. We denote a vertex of a square HEX board which corresponds to a profile (p_k^i, p_{-k}^j) simply by (p_k^i, p_{-k}^j). If individual k is a dictator, the following vertices are marked be white circles.

$$(p_k^1, p_{-k}^3),\ (p_k^1, p_{-k}^4),\ (p_k^1, p_{-k}^5),\ (p_k^1, p_{-k}^6),\ (p_k^2, p_{-k}^3),\ (p_k^2, p_{-k}^4),\ (p_k^2, p_{-k}^5),\ (p_k^2, p_{-k}^6)$$
$$(p_k^3, p_{-k}^1),\ (p_k^3, p_{-k}^2),\ (p_k^3, p_{-k}^5),\ (p_k^3, p_{-k}^6),\ (p_k^4, p_{-k}^1),\ (p_k^4, p_{-k}^2),\ (p_k^4, p_{-k}^5),\ (p_k^4, p_{-k}^6)$$
$$(p_k^5, p_{-k}^1),\ (p_k^5, p_{-k}^2),\ (p_k^5, p_{-k}^3),\ (p_k^5, p_{-k}^4),\ (p_k^6, p_{-k}^1),\ (p_k^6, p_{-k}^2),\ (p_k^6, p_{-k}^3),\ (p_k^6, p_{-k}^4)$$

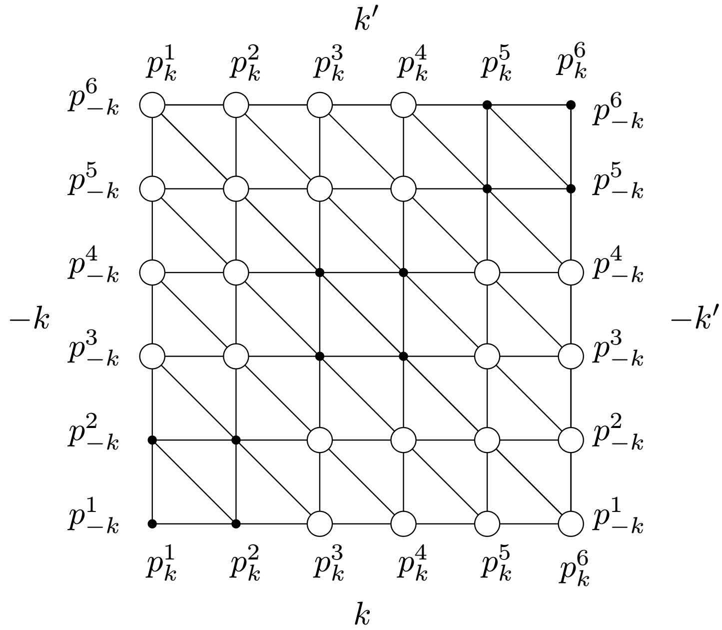


Figure 5: HEX game won by individual k

Then, we obtain Figure 5. Unmarked vertices, where the most preferred alternatives of all individuals are the same, should be randomly marked. Clearly individual k is the winner of this HEX game. Thus, the existence of a dictator for a social choice correspondence implies the existence of a winner for a HEX game. Therefore, we obtain

Theorem 7.3 The Duggan-Schwartz theorem and the HEX game theorem are equivalent.

7.5 Concluding Remarks

We have considered the relationship between the HEX game theorem and the Duggan-Schwartz theorem, and have shown their equivalence. We think that the idea of this chapter can be applied to other social choice theorems which argue the existence of a dictator for some social choice rules.

Chapter 8

The HEX game theorem and the Arrow impossibility theorem: the case of weak orders

We will show that under some assumptions about marking rules of HEX games, the Arrow impossibility theorem that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator when individual preferences are weak orders is equivalent to the HEX game theorem that any HEX game has one winner. Because Gale (1979) showed that the Brouwer fixed point theorem is equivalent to the HEX game theorem, this chapter indirectly shows the equivalence of the Brouwer fixed point theorem and the Arrow impossibility theorem. In Chichilnisky (1979) she showed the equivalence of her impossibility theorem in topological social choice theory (Chichilnisky (1980)) and the Brouwer fixed point theorem, and Baryshnikov (1993) showed that the impossibility theorem in Chichilnisky (1980) and the Arrow impossibility theorem are very similar. Thus, Chichilnisky (1979), (1980) and Baryshnikov (1993) are precedents for the result – linking the Arrow impossibility theorem to a fixed point theorem[*1].

[*1] This chapter is based on my paper of the same title which will be published in *Metroeconomica*, Blackwell.

8.1 Introduction

Gale (1979) showed that the so called HEX game theorem that any HEX game has one winner is equivalent to the Brouwer fixed point theorem. In this chapter we will show that under some assumptions about marking rules of HEX games, the Arrow impossibility theorem that there exists no binary social choice rule which satisfies transitivity, Pareto principle, independence of irrelevant alternatives and has no dictator when individual preferences are weak orders is equivalent to the HEX game theorem.

In another paper Tanaka (2007a) we showed the equivalence of the HEX game theorem and the Arrow impossibility theorem in the case where individual preferences are strong (or linear) orders, that is, individuals are never indifferent about any pair of two alternatives. In that paper we used a 6×6 HEX game. On the other hand in this chapter we will show the equivalence in the case where individual preferences are weak orders, that is, individuals may be indifferent about any pair of two alternatives. And in this chapter we will use a 13×13 HEX game. The result of this chapter is not directly derived from the result of the previous paper. The chief differences are as follows.

> In the proof of Theorem 8.2 we consider the social preference, properly speaking, the society's most preferred alternative at the following profiles of individual preferences about three alternatives x_1, x_2 and x_3.
>
> $$(p_k^7, p_{-k}^4) = (\overline{12}3, 321),\ (p_k^8, p_{-k}^3) = (1\overline{23}, 312),\ (p_k^9, p_{-k}^4) = (\overline{13}2, 321)),$$
> $$(p_k^{12}, p_{-k}^4) = (2\overline{13}, 321)),\ (p_k^{10}, p_{-k}^5) = (3\overline{12}, 231)),\ (p_k^9, p_{-k}^5) = (\overline{13}2, 231)),$$
> $$(p_k^9, p_{-k}^3) = (\overline{13}2, 312)),\ (p_k^{11}, p_{-k}^4) = (\overline{23}1, 321)),\ (p_k^{13}, p_{-k}^4) = (\overline{123}, 321))$$
>
> (p_k^7, p_{-k}^4) denotes a profile such that one individual (denoted by k) prefers x_1 to x_3, prefers x_2 to x_3 and he is indifferent between x_1 and x_2, and all other individuals (denoted by $-k$) prefer x_3 to x_2 to x_1. (p_k^{12}, p_{-k}^4) denotes a profile such that individual k prefers x_2 to x_1, prefers x_2 to x_3 and he is indifferent between x_1 and x_3, and all other individuals prefer x_3 to x_2 to x_1, and so on. The social preferences at these profiles are determined by the social preferences at other profiles such that the preferences of all individuals are strong orders, and the conditions of transitivity, Pareto principle and independence of irrelevant alternatives.

Because Gale (1979) showed that the Brouwer fixed point theorem is equivalent to the HEX game theorem, this chapter indirectly shows the equivalence of the Brouwer fixed point theorem and the Arrow impossibility theorem. In another paper Tanaka (2006a) we directly showed this equivalence using a model according to Baryshnikov (1993). But in that paper we used techniques of algebraic topology (homology theory). Topological approaches to social choice problems have been initiated by Chichilnisky (1979) and (1980). In her model a space of alternatives is a subset of Euclidean space, and individual preferences over this set are represented by normalized gradient fields. Her main result in Chichilnisky (1980) is an impossibility theorem that there exists no *continuous* social choice rule which satisfies *unanimity* and *anonymity*. Unanimity is weaker than Pareto principle, and anonymity is stronger than the condition of non-existence of dictator. In Chichilnisky (1979) she showed the equivalence of her impossibility theorem and the Brouwer fixed point theorem in the case where there are two individuals and the choice space is a subset of 2-dimensional Euclidian space. Baryshnikov (1993) presented a topological approach to the Arrow impossibility theorem in a discrete framework of social choice, and showed that the impossibility

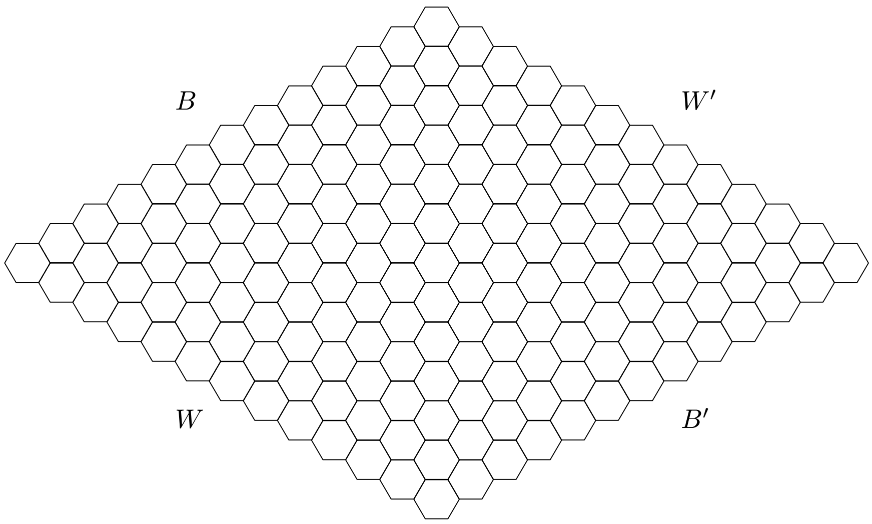


Figure 1: HEX game

theorem in Chichilnisky (1980) and the Arrow impossibility theorem are very similar from the point of view of algebraic topology. Thus, Chichilnisky (1979), (1980) and Baryshnikov (1993) are precedents for the result – linking the Arrow impossibility theorem to a fixed point theorem. See also Chichilnisky (1993) in which her main results are summarized.

In the next section according to Gale (1979) we present an outline of the HEX game. In Section 8.3 we will show that the HEX game theorem implies the Arrow impossibility theorem when individual preferences are weak orders. And in Section 8.4 we will show that the Arrow impossibility theorem implies the HEX game theorem.

8.2 The HEX game

According to Gale (1979) we present an outline of the HEX game. Figure 1 represents a 13 × 13 HEX board. Generally a HEX game is represented by an $n \times n$ HEX board where n is a finite positive integer. The rules of the game are as follows. Two players (called Mr. W and Mr. B) move alternately, marking any previously unmarked hexagon or tile with a white (by Mr. W) or a black (by Mr. B) circle respectively. The game has been won by Mr. W (or Mr. B) if he has succeeded in marking a connected set of tiles which meets the boundary regions W and W' (or, B and B'). A set S of tiles is connected if any two members of the set h and h' can be joined by a path $P = (h = h^1, h^2, \dots, h^m = h')$ where h^i and h^{i+1} are adjacent. Figure 2 represents a HEX game which has been won by Mr. B.

About the HEX game Gale (1979) has shown the following theorem.

Theorem 8.1 (The HEX game theorem) If every tile of the HEX board is marked by either a white or a black circle, then there is a path connecting regions W and W', or a path connecting regions B and B'.

Actually he has shown the theorem that any hex game can never end in a draw, and there always exists at least one winner. But, from his intuitive explanation using the following example of river and dam, it is clear that there exists only one winner of any hex game.

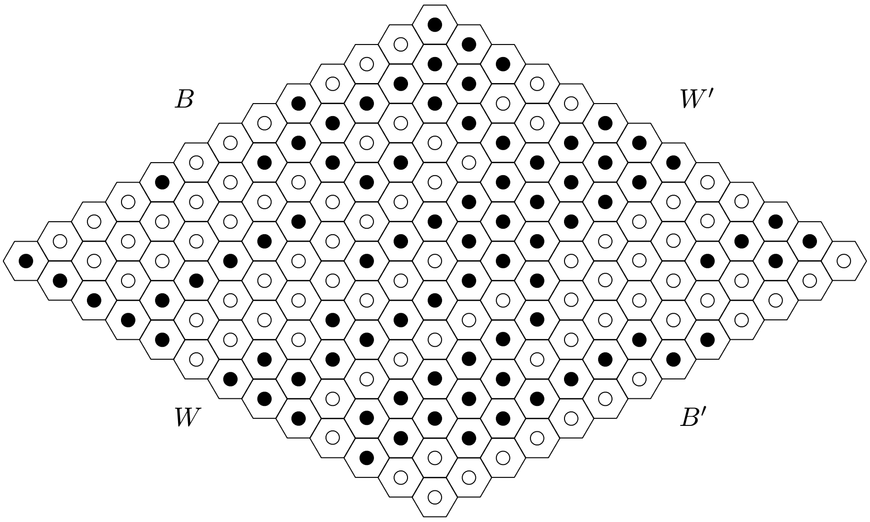


Figure 2: HEX game won by Mr. B

> Imagine that B and B' regions are portions of opposite banks of the river which flow from W region to W' region, and that Mr. B is trying to build a dam by putting down stones. He will have succeeded in damming the river if and only if he has placed his stones in a way which enables him to walk on them from one bank (B region) to the other (B' region).

The proof of Theorem 8.1 and also the above intuitive argument do not depend on the rule "two players move *alternately*". Therefore, this theorem is valid for any marking rule.

Figure 3 is obtained by plotting the center of each hexagon, and connecting these centers by lines. Rotating this graph 45° in anticlockwise direction, we obtain Figure 4. It is an equivalent representation of the HEX board depicted in Figure 1. W and W' represent the regions of Mr. W, and B and B' represent the regions of Mr. B. We call it a square HEX board, and call a game represented by a square HEX board a *square HEX game*. In Figure 4 we depict an example of winning marking by Mr. B. It corresponds to the marking pattern in Figure 2. A set of marked vertices which represents one player's victory is called a *winning path*.

8.3 The HEX game theorem implies the Arrow impossibility theorem

There are $m(\geq 3)$ alternatives and $n(\geq 2)$ individuals. m and n are finite positive integers. The set of individuals is denoted by N. Denote individual i's preference by p_i. A combination of individual preferences, which is called a *profile*, is denoted by $\mathbf{p}(= (p_1, p_2, \cdots, p_n))$. The set of profiles is denoted by $\mathcal{P}^n$. The alternatives are represented by x_i, $i = 1, 2, \cdots, m$. Individual preferences over the alternatives are weak orders, that is, individuals strictly prefer one alternative to another, or are indifferent between them. We assume the free triple property, that is, for each set of three alternatives individual preferences are never restricted.

We consider a binary social choice rule which determines a social preference corresponding to each profile. Binary social choice rules must satisfy the conditions of *transitivity*, *Pareto principle* and *independence*

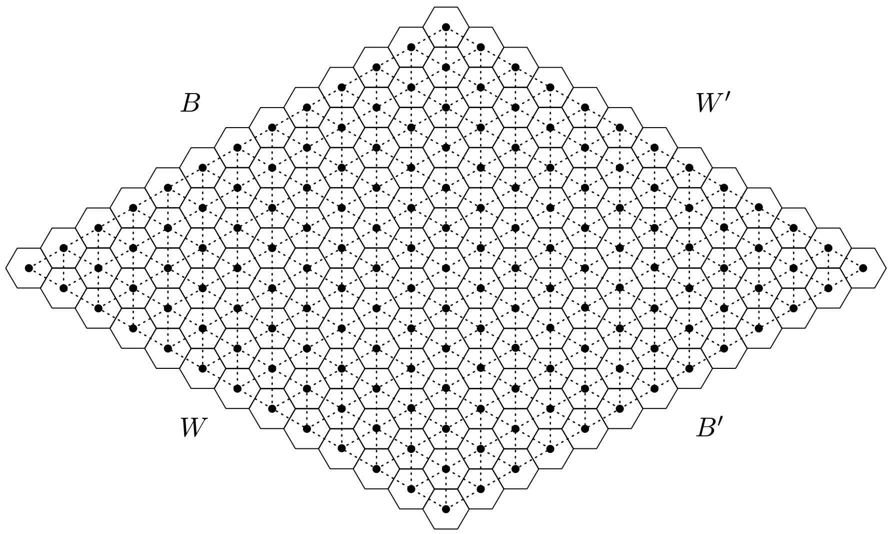


Figure 3: Conversion to square HEX game

of irrelevant alternatives (IIA). Transitive binary social choice rules are called *social welfare functions*. The meanings of these conditions are as follows.

Transitivity If, according to a social welfare function, the society prefers an alternative x_i to another alternative x_j, and prefers x_j to another alternative x_k, then the society must prefer x_i to x_k.

Pareto principle When all individuals prefer x_i to x_j, the society must prefer x_i to x_j.

Independence of irrelevant alternatives (IIA) The social preference about every pair of two alternatives x_i and x_j is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x_i and x_j.

The Arrow impossibility theorem states that there exists no social welfare function which satisfies Pareto principle and IIA and has no dictator, or in other words there exists a dictator for any social welfare function satisfying Pareto principle and IIA. A dictator is an individual whose strict preference always coincides with the social preference.

According to Sen (1979) we define the following terms.

Almost decisiveness If, when all individuals in a group G prefer an alternative x_i to another alternative x_j, and the other individuals (individuals in $N \setminus G$) prefer x_j to x_i, the society prefers x_i to x_j, then G is *almost decisive* for x_i against x_j.

Decisiveness If, when all individuals in a group G prefer an alternative x_i to another alternative x_j, the society prefers x_i to x_j regardless of the preferences of the other individuals, then G is *decisive* for x_i against x_j.

G may consists of one individual. By Pareto principle N is almost decisive and decisive about every pair of alternatives. If for a social welfare function an individual is decisive about every pair of alternatives, then he is the *dictator* of the social welfare function.

Sen (1979)(Lemma 3*a) and Suzumura (2000)(Dictator Lemma) have shown the following result.

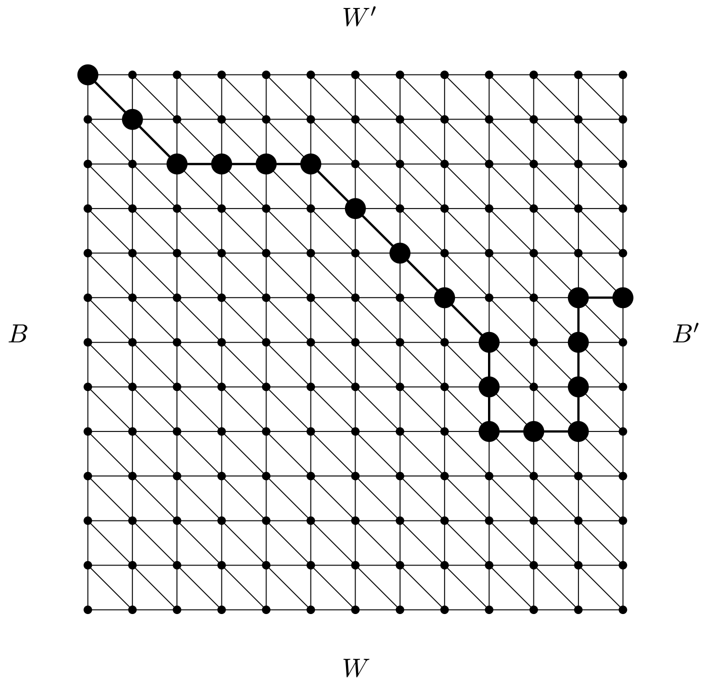


Figure 4: Square HEX game with a winning path

Lemma 8.1 If one individual is almost decisive for one alternative against another alternative, then he is the dictator of the social welfare function.

This lemma holds under the conditions of transitivity, Pareto principle and IIA. The conclusion of this lemma is also valid in the case where not an individual but a group of individuals is almost decisive for one alternative against another alternative. Thus, the following lemma is derived.

Lemma 8.2 If a group of individuals G is almost decisive for one alternative against another alternative, then this group is decisive about every pair of alternatives.

Now we confine us to a subset of profiles $\bar{\mathcal{P}}^n$ such that all individuals prefer three alternatives x_1, x_2 and x_3 to all other alternatives. Pareto principle implies that at all such profiles the society also prefers x_1, x_2 and x_3 to all other alternatives. We denote individual preferences about x_1, x_2 and x_3 in this subset of profiles as follows.

$$p^1 = (123),\ p^2 = (132),\ p^3 = (312),\ p^4 = (321),\ p^5 = (231),\ p^6 = (213),$$
$$p^7 = (\overline{12}3),\ p^8 = (1\overline{23}),\ p^9 = (\overline{13}2),\ p^{10} = (3\overline{12}),\ p^{11} = (\overline{23}1),\ p^{12} = (2\overline{13}),$$
$$p^{13} = (\overline{123})$$

$p^1 = (123)$ represents all preferences such that an individual prefers x_1 to x_2 to x_3 to all other alternatives, $p^1 = (1\overline{23})$ represents all preferences such that an individual prefers x_1 to x_2 and x_3 to all other alternatives and is indifferent between x_2 and x_3, $p^1 = (\overline{13}2)$ represents all preferences such that an individual prefers x_1 and x_3 to x_2 to all other alternatives and is indifferent between x_1 and x_3, and so on. $p^1 = (\overline{123})$ represents a preference such that an individual is indifferent about x_1, x_2 and x_3. Although we confine our arguments to such a subset of profiles, Lemma 8.1 with IIA ensures that an individual who is almost

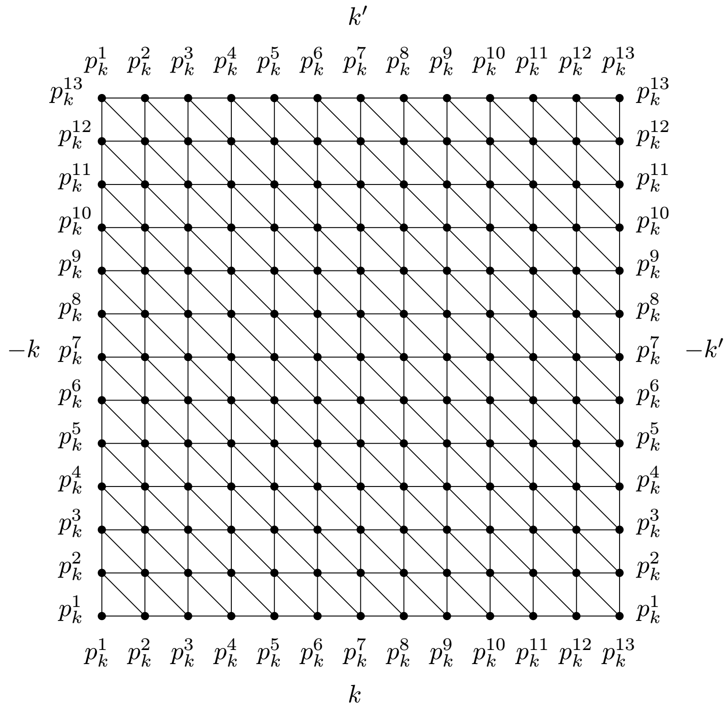


Figure 5: HEX game representing profiles

decisive about a pair of alternatives for this subset of profiles is the dictator for all profiles.

From Lemma 8.2 for the profiles in $\bar{\mathcal{P}}^n$ we obtain the following result.

Lemma 8.3 If two groups G and G', which are not disjoint, are almost decisive about a pair of alternatives, then their intersection $G \cap G'$ is decisive about every pair of alternatives.

Proof. By Lemma 8.2 G and G' are decisive about every pair of alternatives. For three alternatives x_1, x_2 and x_3 we consider the following profile in $\bar{\mathcal{P}}^n$.

1. Individuals in $G \setminus (G \cap G')$ prefer x_3 to x_1 to x_2.
2. Individuals in $G' \setminus (G \cap G')$ prefer x_2 to x_3 to x_1.
3. Individuals in $G \cap G'$ prefer x_1 to x_2 to x_3.
4. Individuals in $N \setminus (G \cup G')$ prefer x_3 to x_2 to x_1.

By the decisiveness of G and G' and transitivity the society must prefer x_1 to x_2 to x_3. Since only individuals in $G \cap G'$ prefer x_1 to x_3 and all other individuals prefer x_3 to x_1, $G \cap G'$ is almost decisive for x_1 against x_3 under IIA. From Lemma 8.2 it is decisive about every pair of alternatives. □

Further we confine us to a subset of $\bar{\mathcal{P}}^n$ such that all but one individual have the same preferences, and consider a HEX game between one individual (denoted by individual k) and the set of individuals other than k. Representative profiles are denoted by (p_k^i, p_{-k}^j), $i = 1, \ldots, 13$, $j = 1, \ldots, 13$, where p_k^i is individual k's preference and p_{-k}^j denotes the common preference of the individuals other than k. We relate these profiles to the vertices of a 13×13 square HEX board as depicted in Figure 5. There are 169 vertices in this HEX board. It represents a square HEX game. k and k' represent individual k's regions, and $-k$ and $-k'$ represent the regions of the set of individuals other than k.

We consider the following marking and winning rules of the square HEX game.

1. At a profile represented by a vertex of a square HEX board, if the society's *unique* most preferred alternative is the same as that of individual k and different from that of the individuals other than k, or that of the individuals other than k is not unique, then this vertex is marked by a white circle; conversely if the society's unique most preferred alternative is the same as that of the individuals other than k and different from that of individual k, or that of individual k is not unique, then this vertex is marked by a black circle.
   Hereafter we abbreviate the most preferred alternative by MPA.
2. The vertex which corresponds to any other profile is randomly marked by a white or a black circle.
3. The game has been won by individual k (or the set of individuals other than k) if he has (or they have) succeeded in marking a connected set of vertices which meets the boundary regions k and k' (or $-k$ and $-k'$).

As a preliminary result we show.

Lemma 8.4 (Lemma 1 in Baryshnikov (1993)) If the individual preferences about x_1 , x_2 and x_3 are strong (linear) orders, that is, the individuals are not indifferent about any pair of theses alternatives, then the society's preference about these alternatives are also strong order.

Proof. Assume that at a profile $\mathbf{p}$ individual k prefers x_1 to x_2 and the other individuals prefer x_2 to x_1 and the society is indifferent between them. Suppose that at a profile $\mathbf{p}'$ individual k prefers x_1 to x_2 to x_3 and the other individuals prefer x_2 to x_3 to x_1, and at a profile $\mathbf{p}''$ individual k prefers x_1 to x_3 to x_2 and the other individuals prefer x_3 to x_2 to x_1. By Pareto principle and IIA the society should prefer x_1 and x_2 to x_3 and should be indifferent between x_1 and x_2 at $\mathbf{p}'$, and it should prefer x_3 to x_1 and x_2 and should be indifferent between x_1 and x_2 at $\mathbf{p}''$. Thus the ranking of x_1 and x_3 depends on the position of x_2 in the preferences of individuals. It is a contradiction.

We can show other cases by similar procedures. □

From this lemma when the individual preferences about x_1 , x_2 and x_3 are strong orders, the society's preference about these alternatives is also strong order.

A square HEX game is equivalent to the original HEX game. Therefore, there exists one winner for any marking rule. Now we show the following result.

Theorem 8.2 The HEX game theorem implies the existence of a dictator for any social welfare function.

Proof. Since diagonal vertices are not connected, the diagonal path

$$((p_k^1, p_{-k}^1), (p_k^2, p_{-k}^2), \cdots, (p_k^{13}, p_{-k}^{13}))$$

can not be a winning path. Consider a profile $(p_k^1, p_{-k}^6) = ((123), (213))$. By Pareto principle x_3 is not the society's MPA. Suppose that at this profile the society's MPA is x_2 which is the MPA of the individuals other than k. Then, by Pareto principle and IIA the society's MPA at the following profiles is x_2.

$$(p_k^1, p_{-k}^5),\ (p_k^2, p_{-k}^6)$$

The fact that the society's MPA at a profile (p_k^2, p_{-k}^6) is x_2, with Pareto principle and IIA, implies that the society's MPA at the following profiles is x_2.

$$(p_k^3, p_{-k}^5),\ (p_k^4, p_{-k}^5),\ (p_k^4, p_{-k}^6)$$

Similarly consider a profile $(p_k^3, p_{-k}^2) = ((312), (132))$. By Pareto principle x_2 is not the society's MPA. Suppose that at this profile the society's MPA is x_1 which is the MPA of the individuals other than k. Then, by Pareto principle and IIA the society's MPA at the following profiles is x_1.

$$(p_k^3, p_{-k}^1),\ (p_k^4, p_{-k}^2)$$

The fact that the society's MPA at a profile (p_k^4, p_{-k}^2) is x_1, with Pareto principle and IIA, implies that the society's MPA at the following profiles is x_1.

$$(p_k^5, p_{-k}^1),\ (p_k^6, p_{-k}^1),\ (p_k^6, p_{-k}^2)$$

Similarly consider a profile $(p_k^5, p_{-k}^4) = ((231), (321))$. By Pareto principle x_1 is not the society's MPA. Suppose that at this profile the society's MPA is x_3 which is the MPA of the individuals other than k. Then, by Pareto principle and IIA the society's MPA at the following profiles is x_3.

$$(p_k^5, p_{-k}^3),\ (p_k^6, p_{-k}^4)$$

The fact that the society's MPA at a profile (p_k^6, p_{-k}^4) is x_3, with Pareto principle and IIA, implies that the society's MPA at the following profiles is x_3.

$$(p_k^1, p_{-k}^3),\ (p_k^2, p_{-k}^3),\ (p_k^2, p_{-k}^4)$$

The results noted above mean that the society's preference about any pair of alternatives among x_1, x_2 and x_3 coincides with the common preference of the individuals other than k when the individual preferences do not include indifference. Then, by Pareto principle and IIA the society's MPA at the following profiles is x_3.

$$(p_k^7, p_{-k}^4) = (\overline{12}3, 321),\ (p_k^8, p_{-k}^3) = (1\overline{23}, 312),\ (p_k^9, p_{-k}^4) = (\overline{13}2, 321)),$$
$$(p_k^{12}, p_{-k}^4) = (2\overline{13}, 321))$$

And the society's MPA at the following profiles is x_2.

$$(p_k^{10}, p_{-k}^5) = (3\overline{12}, 231)),\ (p_k^9, p_{-k}^5) = (\overline{13}2, 231))$$

Further, these results imply that the society's MPA at the following profiles is x_3.

$$(p_k^9, p_{-k}^3) = (\overline{13}2, 312)),\ (p_k^{11}, p_{-k}^4) = (\overline{23}1, 321)),\ (p_k^{13}, p_{-k}^4) = (\overline{12}3, 321))$$

The vertices which correspond to all of these profiles are marked by black circles. Then, we obtain a marking pattern of a square HEX board as depicted in Figure 6. The set of individuals other than k is the winner of this HEX game. Therefore, for individual k to be the winner of a square HEX game, the society's MPA must coincide with that of individual k at least at one of three profiles (p_k^1, p_{-k}^6), (p_k^3, p_{-k}^2) and (p_k^5, p_{-k}^4). It means that individual k must be almost decisive about at least one pair of alternatives, and then by Lemma 8.1 he is the dictator.

If for all k, $(k = 1, 2, \cdots, n-1)$, individual k is not the winner of any square HEX game between individual k and the set of individuals other than k, then each set of individuals excluding one individual is the winner of each square HEX game. By Lemma 8.3 every nonempty intersection of the sets of individuals excluding one individual (among 1 to $n-1$) is decisive. Then, the intersection of $N \setminus \{1\}$, $N \setminus \{2\}$, $\cdots$, $N \setminus \{n-1\}$ is decisive. But $(N \setminus \{1\}) \cap (N \setminus \{2\}) \cap \cdots (N \setminus \{n-1\}) = \{n\}$. Thus, individual n is the dictator. Therefore, the HEX game theorem implies the existence of a dictator for any social welfare function. □

By this theorem the HEX game theorem implies the Arrow impossibility theorem.

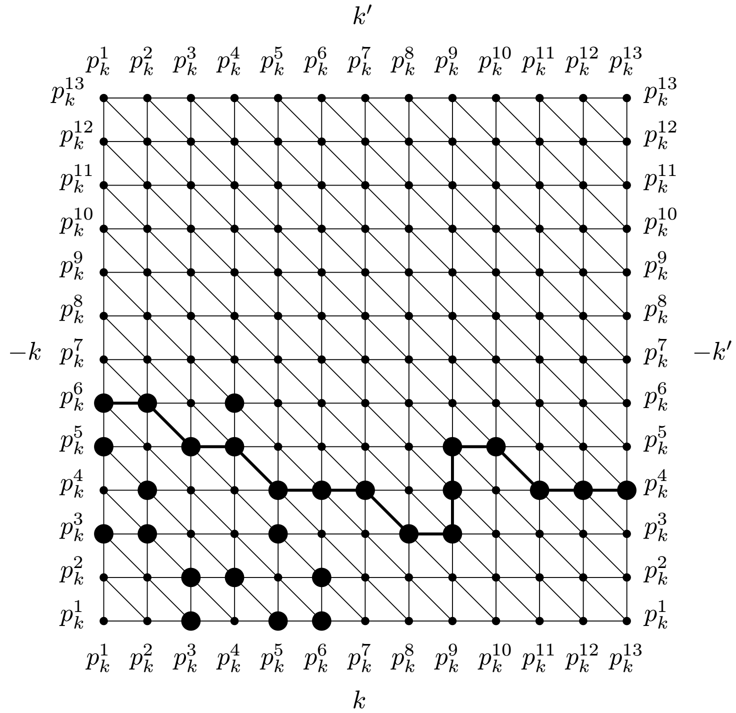


Figure 6: Winning path of a square HEX game

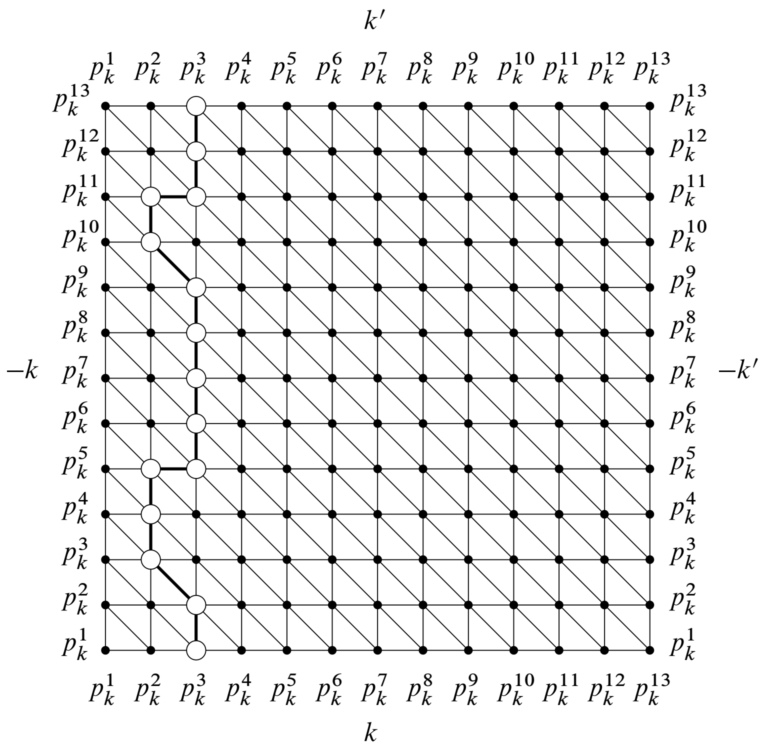


Figure 7: HEX game won by individual k

8.4 The Arrow impossibility theorem implies the HEX game theorem

Next we will show that the Arrow impossibility theorem implies the HEX game theorem under an interpretation of dictator. Similarly to the previous section, we confine us to a subset of profiles such that all individuals prefer three alternatives x_1, x_2 and x_3 to all other alternatives, and the preferences of individuals other than one individual (denoted by k) are the same. And we consider a square HEX game between individual k and the set of individuals other than k. The dictator of a social welfare function is interpreted as an individual who can determine the MPA of the society when his *unique* MPA and that of the other individuals are different, or his MPA is unique and that of the other individuals is not unique, and in a HEX game he can mark tiles with his color in such cases. We denote a vertex of a square HEX board which corresponds to a profile (p_k^i, p_{-k}^j) simply by (p_k^i, p_{-k}^j).

First, consider the case where individual k is the dictator of a social welfare function. Then, the following vertices are marked by white circles.

$$(p_k^2, p_{-k}^3),\ (p_k^2, p_{-k}^4),\ (p_k^2, p_{-k}^5),\ (p_k^3, p_{-k}^1),\ (p_k^3, p_{-k}^2),\ (p_k^3, p_{-k}^5),\ (p_k^3, p_{-k}^6)$$
$$(p_k^3, p_{-k}^7),\ (p_k^3, p_{-k}^8),\ (p_k^3, p_{-k}^9),\ (p_k^3, p_{-k}^{11}),\ (p_k^3, p_{-k}^{12}),\ (p_k^3, p_{-k}^{13}),\ (p_k^2, p_{-k}^{10})$$
$$(p_k^2, p_{-k}^{11})$$

Then, we obtain Figure 7. Clearly individual k is the winner of this HEX game.

Next, consider the case where the dictator of a social welfare function is included in the set of individuals other than k. Then, by symmetric consideration the set of individuals other than k is the winner of the HEX game.

Thus, the existence of a dictator for a social welfare function implies the existence of a winner for a HEX game. Therefore, we obtain

Theorem 8.3 The Arrow impossibility theorem and the HEX game theorem are equivalent.

8.5 Concluding Remarks

We have considered the relationship between the HEX game theorem and the Arrow impossibility theorem when individual preferences are weak orders, and have shown their equivalence.

Chapter 9

Type two computability of social choice functions and the Gibbard-Satterthwaite theorem in an infinite society

This chapter investigates the computability problem of the Gibbard-Satterthwaite theorem (Gibbard (1973), Satterthwaite (1975)) of social choice theory in a society with an infinite number of individuals (infinite society) based on Type two computability by Weihrauch (1995), Weihrauch (2000). There exists a dictator or there exists no dictator for any coalitionally strategy-proof social choice function in an infinite society. We will show that if there exists a dictator for a social choice function, it is computable in the sense of Type two computability, but if there exists no dictator it is not computable. A dictator of a social choice function is an individual such that if he strictly prefers an alternative (denoted by x) to another alternative (denoted by y), then it does not choose y, and his most preferred alternative is always chosen. Coalitional strategy-proofness is an extension of the ordinary strategy-proofness. It requires non-manipulability by coalitions of individuals as well as by a single individual[*1].

9.1 Introduction

This chapter investigates the computability problem of the Gibbard-Satterthwaite theorem (Gibbard (1973), Satterthwaite (1975)) of social choice theory in a society with an infinite number of individuals (infinite society) based on Type two computability by Weihrauch (1995), Weihrauch (2000). Arrow's impossibility theorem Arrow (1963) shows that, with a finite number of individuals, for any social welfare function (binary social choice rule which satisfies transitivity) there exists a dictator. In contrast Fishburn (1970), Hansson (1976) and Kirman and Sondermann (1972) show that in a society with an infinite number of individuals (an infinite society), there exists a social welfare function without dictator. On the other hand, about strategy-proof social choice functions, with a finite number of individuals, the Gibbard-Satterthwaite theorem (Gibbard (1973), Satterthwaite (1975)) shows that there exists a dictator for any strategy-proof social choice function. In contrast Pazner and Wesley (1977) shows that in

[*1] This chapter is based on my paper of the same title published in *Applied Mathematics and Computation*, Vol. 192, No. 1, pp. 168-174, 2007, Elsevier.

an infinite society, there exists a coalitionally strategy-proof social choice function without dictator*2. A dictator of a social choice function is an individual such that if he strictly prefers an alternative (denoted by x) to another alternative (denoted by y), then it does not choose y, and it chooses his most preferred alternative. Coalitional strategy-proofness is an extension of the ordinary strategy-proofness. It requires non-manipulability by coalitions of individuals as well as by a single individual.

In the next section we present the framework of this chapter and some preliminary results. In Section 9.3 we will show the following results.

1. There exists a dictator or there exists no dictator for any coalitionally strategy-proof social choice function, and in the latter case all co-finite sets of individuals (sets of individuals whose complements are finite) are decisive sets (Theorem 9.1).
2. If there exists a dictator, the social choice function is computable in the sense of Type two computability, but if there exists no dictator it is not computable (Theorem 9.2).

A decisive set for a social choice function is a set of individuals such that if individuals in the set prefer an alternative (denoted by x) to another alternative (denoted by y), then the social choice function does not choose y regardless of the preferences of other individuals.

Mihara (1997) presented an analysis about the ordinary Turing machine computability of social choice rules. Since there are only countable number of ordinary Turing machines, he assumes that only countable number of profiles of individual preferences are observable. But Type two machine can treat uncountable input.

9.2 The framework and preliminary results

There are $m(\geq 3)$ alternatives and a countably infinite number of individuals. m is a finite positive integer. The set of alternatives is denoted by A. The set of individuals is denoted by N. The alternatives are represented by x, y, z, w and so on. Individual preferences over the alternatives are transitive linear (strict) orders, that is, they prefer one alternative to another alternative, and are not indifferent between them. Denote individual i's preference by $\succ_i$. We denote $x \succ_i y$ when individual i prefers x to y. Since there are a finite number of alternatives, the varieties of linear orders over the alternatives are finite. We denote the set of individual preferences by Σ. A combination of individual preferences, which is called a *profile*, is denoted by $p(= (\succ_1, \succ_2, \cdots))$, $p'(= (\succ'_1, \succ'_2, \cdots))$ and so on. The set of profiles is denoted by Σ^ω, where $\omega = \{1, 2, \cdots\}$ is the set of natural numbers. It represents the set of individuals.

We consider a social choice function $f : \Sigma^\omega \longrightarrow A$ which chooses at least one and at most one alternative corresponding to each profile of the revealed preferences of individuals. We require that social choice functions are *coalitionally strategy-proof*. This means that any group (coalition) of individuals can not benefit by revealing preferences which are different from their true preferences, in other words, each coalition of individuals must have an incentive to reveal their true preferences, and cannot manipulate any social choice function. The coalitional strategy-proofness is an extension of the ordinary strategy-proofness which requires only non-manipulability by an individual. We also require that social choice functions are *onto*, that is, their ranges are A. The Gibbard-Satterthwaite theorem states that, with a finite number of individuals, there exists a dictator for any strategy-proof social choice function, or in other

*2 Taylor (2005) is a recent book that discusses social choice problems in an infinite society.

words there exists no social choice function which satisfies strategy-proofness and has no dictator. In contrast Pazner and Wesley (1977) shows that when the number of individuals in the society is infinite, there exists a coalitionally strategy-proof social choice function without dictator. A dictator of a social choice function is an individual whose most preferred alternative is always chosen by the social choice function.

According to Weihrauch (1995), Weihrauch (2000) we survey the definitions of Type two machine and Type two computability, and consider the formulation of a social choice function computed by a Type two machine.

Type two machine A Type two machine M (with one input tape) is defined by two components.

1. A Turing machine with a single one-way input tape, a single one-way output tape and finitely many work tapes.
2. A type specification (Y_1, Y_0) with $\{Y_1, Y_0\} \in \{\Sigma^*, \Sigma^\omega\}$. Σ denotes any finite alphabet. $\omega = \{1, 2, \cdots\}$ is the set of natural numbers. Σ^* is the set of all finite sequences $\succ_1\succ_2 \cdots \succ_k$ with $k \in \omega$ and $\succ_1, \succ_2, \cdots, \succ_k \in \Sigma$. And $\Sigma^\omega = \{\succ_1\succ_2 \cdots \mid \succ_i \in \Sigma\} = \{p | p : \omega \longrightarrow \Sigma\}$ is the set of infinite sequences with elements from Σ.

   The function $f_M : Y_1 \longrightarrow Y_0$ computed by a Type two machine M is defined as follows:

   (a) Case $Y_0 = \Sigma^*$(finite output)

   $f_M(y_1) = w \in \Sigma^*$ if and only if M with input (y_1) halts with result w on the output tape.

   (b) Case $Y_0 = \Sigma^\omega$(infinite output)

   $f_M(y_1) = p \in \Sigma^\omega$ if and only if M with input (y_1) computes forever writing the sequence p on the output tape.

Type two computability Let Σ be a finite alphabet. Assume $Y_1 \subseteq \{\Sigma^*, \Sigma^\omega\}$. A function $f : Y_1 \longrightarrow Y_0$ is computable if and only if $f = f_M$ for some Type two machine M.

A social choice function computed by a Type two machine A social choice function is defined as a function $f : \Sigma^\omega \longrightarrow A$. Σ^ω is the set of profiles, and A is the set of alternatives as alphabets. An element of Σ^ω, $p \in \Sigma^\omega$, is a profile, and an element of A is an alternative.

Now we define the following terms[*3].

Monotonicity Let x and y be two alternatives. Assume that at a profile p individuals in a group G prefer x to y, all other individuals (individuals in $N \setminus G$) prefer y to x, and x is chosen by a social choice function. If at another profile p' individuals in G prefer x to y, then the social choice function does not choose y regardless of the preferences of the individuals in $N \setminus G$.

Weak Pareto principle If all individuals prefer x to y, then any social choice function does not choose y.

First we can show the following lemma.

Lemma 9.1 If a social choice function satisfies coalitional strategy-proofness, then it satisfies monotonicity and weak Pareto principle.

Proof. See Section 9.5. □

[*3] The concept *monotonicity* is according to Batteau,Blin (and Monjardet). It is equivalent to *strong positive association* by Muller and Satterthwaite (1975) when individual preferences are linear orders (do not include indifference relations).

Further we define the following two terms.

Decisive If, when all individuals in a group G prefer an alternative x to another alternative y, a social choice function does not chooses y regardless of the preferences of other individuals, then G is *decisive* for x against y.

Decisive set If a group of individuals is decisive about every pair of alternatives for a social choice function, it is called a decisive set for the social choice function.

The meaning of the term *decisive* is similar to that of the same term used in Sen (1979) for binary social choice rules. G may consist of one individual. If for a social choice function an individual is decisive about every pair of alternatives, then he is a *dictator* of the social choice function.

About the concept of decisiveness we can show the following result.

Lemma 9.2 Assume that a social choice function is coalitionally strategy-proof. If a group G is decisive for one alternative against another alternative, then it is a decisive set.

Proof. See Section 9.6. □

The implications of this lemma are similar to those of Lemma 3*a in Sen (1979) and Dictator Lemma in Suzumura (2000) for binary social choice rules.

Next we can show the following result.

Lemma 9.3 Assume that a social choice function is coalitionally strategy-proof. If two groups G and G', which are not disjoint, are decisive sets, then their intersection $G \cap G'$ is a decisive set.

Proof. Let x, y and z be given three alternatives, and consider the following profile.

1. Individuals in $G \setminus (G \cap G')$ prefer z to x to y to all other alternatives.
2. Individuals in $G' \setminus (G \cap G')$ prefer y to z to x to all other alternatives.
3. Individuals in $G \cap G'$ prefer x to y to z to all other alternatives.
4. Individuals in $N \setminus (G \cup G')$ prefer z to y to x to all other alternatives.

Since G and G' are decisive sets, the social choice function chooses x. Only individuals in $G \cap G'$ prefer x to z and all other individuals prefer z to x. Thus, by monotonicity $G \cap G'$ is decisive for x against z. By Lemma 9.2 it is a decisive set. □

Note that G and G' can not be disjoint. Assume that G and G' are disjoint. If individuals in G prefer x to y to all other alternatives, and individuals in G' prefer y to x to all other alternatives, then neither G nor G' can be a decisive set.

This lemma implies that the intersection of a finite number of decisive sets is also a decisive set.

9.3 Type two computability of coalitionally strategy-proof social choice functions

Consider profiles such that one individual (denoted by i) prefers x to y to z to all other alternatives, and all other individuals prefer z to x to y to all other alternatives. Denote such a profile by p^i. By weak Pareto principle any social choice function chooses x or z. If a social choice function chooses x at p^i for some i, then by monotonicity individual i is decisive for x against z, and by Lemma 9.2 he is a dictator. On the other hand, if the social choice function chooses z at p^i for all $i \in N$, then there exists no dictator, and a group $N \setminus \{i\}$ is a decisive set for all $i \in N$. By Lemma 9.3 in the latter case all co-finite sets (sets of individuals whose complements are finite sets) are decisive sets. Thus, we obtain the following theorem.

Theorem 9.1 For any coalitionally strategy-proof social choice function there exists a dictator or there exists no dictator, and in the latter case all co-finite sets are decisive sets.

Let partition the set of individuals into a finite number of groups $G_1, G_2, \cdots, G_k$. Each group may include a finite or an infinite number of individuals. If there exists no dictator, all co-finite sets are decisive sets, and then every finite group is not a decisive set. Therefore, the decisive set must be an infinite group, and we obtain the following result.

Lemma 9.4 Suppose that there exists no dictator for a social choice function. Let partition the set of individuals into a finite number of groups $G_1, G_2, \cdots, G_k$. Each group may include a finite or an infinite number of individuals. Then, one of infinite groups is a decisive set.

Finally we show the following main result of this chapter.

Theorem 9.2
1. If there exists a dictator for a social choice function, then it is computable in the sense of Type two computability.
2. If there exists no dictator for a social choice function, then it is not computable.

Proof.
1. Assume that individual $i \, (\in \omega)$ is a dictator of a social choice function. A Type two machine can determine the choice of the society from the i-th input, and then it halts.
2. Let partition the individuals into a finite number of groups corresponding to the preferences of individuals in each group. Consider a profile $p \in \Sigma^\omega$ such that in such a partition only one group includes an infinite number of individuals. Then, by Lemma 9.4 this group is a decisive set. But any Type two machine can not determine which group is an infinite group in finite steps, and it can not halt. Therefore, the social choice function is not computable.

□

9.4 Concluding Remarks

We have examined the Gibbard-Satterthwaite theorem of social choice theory in an infinite society. The assumption of an infinite society seems to be unrealistic. But Mihara (1997) presented an interpretation of an infinite society based on a *finite* number of individuals and a countably infinite number of uncertain states.

In this chapter we assumed that individual preferences are linear orders, that is, they are not indifferent about any pair of alternatives. In the case of weak orders, which include indifference relations, a social choice function may not be computable even when there exists a dictator. Consider a social choice function such that when its dictator's most preferred alternatives are not unique, the society's choice is determined by preferences of a group of individuals with an infinite number of individuals. Then this social choice function is not computable.

9.5 Proof of Lemma 9.1

We use notations in the definition of monotonicity.

1. (Monotonicity) Let z be an arbitrary alternative other than x and y. Assume that at a profile p'' individuals in G prefer x to y to z, and other individuals prefer y to x to z. If, when the preferences of some individuals in G change from $\succ_i$ (their preferences at p) to $\succ_i''$ (their preferences at p''), x is not chosen by the social choice function, then they can gain benefit by revealing their preferences $\succ_i$ when their true preferences are $\succ_i''$. Thus, the social choice function continues to choose x in this case. By the same logic, when the preferences of all individuals in G change to their preferences at p'', the social choice function chooses x. Next, if, when the preferences of some individuals in $N \setminus G$ change from $\succ_i$ to $\succ_i''$, the social choice function chooses y, then they can gain benefit by revealing their preferences $\succ_i''$ when their true preferences are $\succ_i$. On the other hand, if z is chosen in this case, they can gain benefit by revealing their preferences $\succ_i$ when their true preferences are $\succ_i''$. Thus, x must be chosen. By the same logic, when the preferences of all individuals change to their preferences at p'', the social choice function chooses x.
   Next, if, when the preferences of some individuals in G change from $\succ_i''$ to $\succ_i'$ (their preferences at p'), the alternative chosen by the social choice function changes directly from x to y, then they can gain benefit by revealing their preferences $\succ_i''$ when their true preferences are $\succ_i'$. Thus, the alternative chosen by the social choice function does not directly change from x to y in this case. By the same logic, when the preferences of all individuals in G change to their preferences at p', the alternative chosen by the social choice function does not directly change from x to y. Further, if, when the preferences of some individuals in $N \setminus G$ change from $\succ_i''$ to $\succ_i'$, the alternative chosen by the social choice function changes directly from x to y, then they can gain benefit by revealing their preference $\succ_i'$ when their true preferences are $\succ_i''$. By the same logic, when the preferences of all individuals change to their preferences at p', the alternative chosen by the social choice function does not directly change from x to y.
   There is a possibility, however, that the alternative chosen by the social choice function changes from x through $w(\neq x, y)$ to y in transition from p'' to p'. If, when the preferences of some individuals change, the alternative chosen by the social choice function changes from x to w, and further when the preferences of other some individuals (denoted by i) change, the alternative chosen by the social choice function changes to y, they have incentives to reveal their preferences $\succ_i'$ when their true preferences are $\succ_i''$ because they prefer y to w at p''. Therefore, y is not chosen by the social choice function at p'.
2. (Weak Pareto principle) Let p be a profile at which all individuals prefer x to y, and p' be a profile at which x is chosen by the social choice function. Assume that at another profile p'' all individuals prefer x to y to all other alternatives. If, when the preferences of some individuals change from $\succ_i'$ to $\succ_i''$, the social choice function does not choose x, then they can gain benefit by revealing their preferences $\succ_i'$ when their true preferences are $\succ_i''$. Thus, x is chosen in this case. By the same logic, when the preferences of all individuals change to their preferences at p'', x is chosen. Since at p'' and at p all individuals prefer x to y, monotonicity (proved in (1)) implies that y is not chosen by the social choice function at p.

9.6 Proof of Lemma 9.2

1. Case 1: There are more than three alternatives.
   Assume that G is decisive for x against y. Let z and w be given alternatives other than x and y. Consider the following profile.
   (a) Individuals in G prefer z to x to y to w to all other alternatives.
   (b) Other individuals prefer y to w to z to x to all other alternatives.
   By weak Pareto principle the social choice function chooses y or z. Since G is decisive for x against y, z is chosen. Then, by monotonicity the social choice function does not choose w so long as the individuals in G prefer z to w. It means that G is decisive for z against w. From this result by similar procedures we can show that G is decisive for x (or y) against w, for z against x (or y), and for y against x. Since z and w are arbitrary, G is decisive about every pair of alternatives, that is, it is a decisive set.
2. Case 2: There are only three alternatives x, y and z.
   Assume that G is decisive for x against y. Consider the following profile.
   (a) Individuals in G prefer x to y to z.
   (b) Other individuals prefer y to z to x.
   By weak Pareto principle the social choice function chooses x or y. Since G is decisive for x against y, x is chosen. Then, by monotonicity the social choice function does not choose z so long as the individuals in G prefer x to z. It means that G is decisive for x against z. Similarly we can show that G is decisive for z against y considering the following profile.
   (a) Individuals in G prefer z to x to y.
   (b) Other individuals prefer y to z to x.
   By similar procedures we can show that G is decisive for y against z, for z against x, and for y against x.

Chapter 10

The Arrow impossibility theorem of social choice theory in an infinite society and LPO (Limited principle of omniscience)

This chapter is an attempt to examine the main theorems of social choice theory from the viewpoint of constructive mathematics. We examine the Arrow impossibility theorem (Arrow (1963)) in a society with an infinite number of individuals (infinite society). We will show that the theorem that there exists a dictator or there exists no dictator for any binary social choice rule satisfying transitivity, Pareto principle and independence of irrelevant alternatives in an infinite society is equivalent to LPO (Limited principle of omniscience). Therefore, it is non-constructive. A dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society must also strictly prefer the former to the latter*1.

10.1 Introduction

This chapter is an attempt to examine the main theorems of social choice theory from the viewpoint of constructive mathematics. Arrow's impossibility theorem (Arrow (1963)) shows that, with a finite number of individuals, for any social welfare function (transitive binary social choice rule) which satisfies Pareto principle and independence of irrelevant alternatives (IIA) there exists a dictator. A dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society must also strictly prefer the former to the latter. On the other hand, Fishburn (1970), Hansson (1976) and Kirman and Sondermann (1972) show that, in a society with an infinite number of individuals (infinite society), there exists a social welfare function satisfying Pareto principle and IIA without dictator*2.

In this chapter we will show that the theorem that there exists a dictator or there exists no dictator for any social welfare function satisfying Pareto principle and IIA in an infinite society is equivalent to LPO (Limited principle of omniscience). Therefore, it is non-constructive.

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*1 This chapter is based on my paper of the same title which will be published in *Applied Mathematics E-Notes*, National Tsing Hua University (Taiwan).

*2 Taylor (2005) is a recent book that discusses social choice problems in an infinite society.

The omniscience principles are general statements that can be proved classically but not constructively, and are used to show that other statements do not admit constructive proofs*3. This is done by showing that the statement implies the omniscience principle. The strongest omniscience principle is the law of excluded middle. A weaker one is the following limited principle of omniscience (abbreviated as LPO).

LPO (Limited principle of omniscience) Given a binary sequence a_n, $n \in \mathbb{N}$(the set of positive integers), either $a_n = 0$ for all n or $a_n = 1$ for some n.

In the next section we present the framework of this chapter and some preliminary results. In Section 10.3 we will show the following results.

1. There exists a dictator or there exists no dictator for any social welfare function satisfying Pareto principle and IIA, and in the latter case all co-finite sets of individuals (sets of individuals whose complements are finite) are decisive sets (Theorem 10.1).
2. Theorem 10.1 is equivalent to LPO (Theorem 10.2).

A decisive set is a set of individuals such that if individuals in the set prefer an alternative (denoted by x) to another alternative (denoted by y), then the society prefers x to y regardless of the preferences of other individuals.

10.2 The framework and preliminary results

There are more than two (finite or infinite) alternatives and a countably infinite number of individuals. The set of individuals is denoted by N. The set of alternatives is denoted by A. N and A are discrete sets*4. For each pair of elements i, j of N we have $i = j$ or $i \neq j$, and for each pair of elements x, y of A we have $x = y$ or $x \neq y$. Each subset of N is detachable. Thus, for each individual i of N and each subset I of N we have $i \in I$ or $i \notin I$. The alternatives are represented by x, y, z, w and so on. Denote individual i's preference by $\succ_i$. We denote $x \succ_i y$ when individual i prefers x to y. Individual preferences over the alternatives are transitive weak orders, and they are characterized constructively according to Bridges (1999). About given three alternatives x, y and z individual i's preference satisfies the following properties.

1. If $x \succ_i y$, then $\neg(y \succ_i x)$.
2. If $x \succ_i y$, then for each $z \in A$ either $x \succ_i z$ or $z \succ_i y$.

Preference-indifference relation $\succsim_i$ and indifference relation $\sim_i$ are defined by

- $x \succsim_i y$ if and only if $\forall z \in A(y \succ_i z \Rightarrow x \succ_i z)$,
- $x \sim_i y$ if and only if $x \succsim_i y$ and $y \succsim_i x$.

Then, the following results are derived.

- $\neg(x \succ_i x)$.
- $x \succ_i y$ entails $x \succsim_i y$.

*3 About omniscience principles we referred to Bridges and Richman (1987), Bridges and Vîţă (2006), Mandelkern (1983) and Mandelkern (1989).

*4 About details of the concepts of discrete set and detachable set, see Bridges and Richman (1987).

- The relations $\succ_i$, $\succsim_i$ are transitive, and $x \succsim_i y \succ_i z$ entails $x \succ_i z$.
- $x \succsim_i y$ if and only if $\neg(y \succ_i x)$.

As demonstrated by Bridges (1999) we can not prove constructively that $x \succ_i y$ if and only if $\neg(y \succsim_i x)$.

A combination of individual preferences, which is called a *profile*, is denoted by $\mathbf{p}(= (\succ_1, \succ_2, \cdots))$, $\mathbf{p}'(= (\succ'_1, \succ'_2, \cdots))$ and so on.

We consider a binary social choice rule which determines a social preference corresponding to each profile. Social preferences are defined similarly to individual preferences. We denote $x \succ y$ when the society strictly prefers x to y. The social preference is denoted by $\succ$ at $\mathbf{p}$, by $\succ'$ at $\mathbf{p}'$ and so on, and it satisfies the following conditions.

1. P1: If $x \succ y$, then $\neg(y \succ x)$.
2. P2: If $x \succ y$, then for each $z \in A$ either $x \succ z$ or $z \succ y$.

$x \succsim y$ and $x \sim y$ are defined as follows.

- $x \succsim y$ if and only if $\forall z \in A(y \succ z \Rightarrow x \succ z)$,
- $x \sim y$ if and only if $x \succsim y$ and $y \succsim x$.

Then, the following results are derived.

- $\neg(x \succ x)$
- $x \succ y$ entails $x \succsim y$.
- The relations $\succ$, $\succsim$ are transitive, and $x \succsim y \succ z$ entails $x \succ z$.
- $x \succsim y$ if and only if $\neg(y \succ x)$.

Social preferences are further required to satisfy *Pareto principle* and *independence of irrelevant alternatives (IIA)*. The meanings of these conditions are as follows.

Pareto principle When all individuals prefer x to y, the society must prefer x to y.

Independence of irrelevant alternatives (IIA) The social preference about every pair of two alternatives x and y is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x and y.

A binary social choice rule which satisfies transitivity is called a *social welfare function*. Arrow's impossibility theorem (Arrow (1963)) shows that, with a finite number of individuals, for any social welfare function satisfying Pareto principle and IIA there exists a dictator. In contrast Fishburn (1970), Hansson (1976) and Kirman and Sondermann (1972) show that when the number of individuals in the society is infinite, there exists a social welfare function satisfying Pareto principle and IIA without dictator. A dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society must also strictly prefer the former to the latter.

According to definitions in Sen (1979) we define the following terms.

Almost decisiveness If, when all individuals in a (finite or infinite) group G prefer an alternative x to another alternative y, and other individuals (individuals in $N \setminus G$) prefer y to x, the society prefers x to y ($x \succ y$), then G is *almost decisive* for x against y.

Decisiveness If, when all individuals in a group G prefer x to y, the society prefers x to y regardless of the preferences of other individuals, then G is *decisive* for x against y.

Decisive set If a group of individuals is decisive about every pair of alternatives, it is called a decisive set.

A decisive set may consist of one individual. If an individual is decisive about every pair of alternatives for a social welfare function, then he is a *dictator* of the social welfare function. Of course, there exists at most one dictator.

First about decisiveness we show the following lemma.

Lemma 10.1 If a group of individuals G is almost decisive for an alternative x against another alternative y, then it is decisive about every pair of alternatives, that is, it is a decisive set.

***Proof*.** See Section 10.5.

The implications of this lemma are similar to those of Lemma 3*a in Sen (1979) and Dictator Lemma in Suzumura (2000). Next we show the following lemma.

Lemma 10.2 If G_1 and G_2 are decisive sets, then $G_1 \cap G_2$ is also a decisive set.

***Proof*.** Let x, y and z be given three alternatives, and consider the following profile.

1. Individuals in $G_1 \setminus G_2$ (denoted by i): $z \succ_i x \succ_i y$
2. Individuals in $G_2 \setminus G_1$ (denoted by j): $y \succ_j z \succ_j x$
3. Individuals in $G_1 \cap G_2$ (denoted by k): $x \succ_k y \succ_k z$
4. Other individuals (denoted by l): $z \succ_l y \succ_l x$

Since G_1 and G_2 are decisive sets, the social preference is $x \succ y$ and $y \succ z$. Then, by transitivity the social preference about x and z should be $x \succ z$. Only individuals in $G_1 \cap G_2$ prefer x to z, and all other individuals prefer z to x. Thus, $G_1 \cap G_2$ is almost decisive for x against z. Then, by Lemma 10.1 it is a decisive set.

Note that G_1 and G_2 can not be disjoint. Assume that G_1 and G_2 are disjoint. If individuals in G_1 prefer x to y, and individuals in G_2 prefer y to x, then neither G_1 nor G_2 can be a decisive set.

This lemma implies that the intersection of a finite number of decisive sets is also a decisive set.

10.3 Existence of social welfare function satisfying Pareto principle and IIA without dictator and LPO

Consider profiles such that one individual (denoted by i) prefers x to y to z, and all other individuals prefer z to x to y. Denote such a profile by $\mathbf{p}^i$. By Pareto principle the social preference about x and y is $x \succ y$. By the property of constructively defined social preference (P2) the social preference is $x \succ z$ or $z \succ y$. If it is $x \succ z$ at $\mathbf{p}^i$ for some i, then by IIA individual i is almost decisive for x against z, and by Lemma 10.1 he is a dictator. On the other hand, if the social preference is $z \succ y$ at $\mathbf{p}^i$ for all $i \in N$, then there exists no dictator. In this case by IIA, Lemma 10.1 and 10.2 all co-finite sets (sets of individuals whose complements are finite sets) are decisive sets. Thus, we obtain

Theorem 10.1 For any social welfare function satisfying Pareto principle and IIA there exists a dictator or there exists no dictator, and in the latter case all co-finite sets are decisive sets.

But we can show the following theorem.

Theorem 10.2 Theorem 10.1 is equivalent to LPO.

Proof. Define a binary sequence (a_i) as follows.

$$a_i = 1 \text{ for } i \in \mathbb{N} \text{ if the social preference about } x \text{ and } z \text{ at } \mathbf{p}^i \text{ is } x \succ z$$
$$a_i = 0 \text{ for } i \in \mathbb{N} \text{ if the social preference about } y \text{ and } z \text{ at } \mathbf{p}^i \text{ is } z \succ y$$

The condition of LPO for this binary sequence is as follows.

LPO (Limited principle of omniscience)

$$a_i = 0 \text{ for all } i \in \mathbb{N} \text{ or } a_i = 1 \text{ for some } i \in \mathbb{N}$$

From the arguments before Theorem 10.1 it is clearly equivalent to Theorem 10.1.

Note $x \succ z$ and $z \succ y$ are not consistent at $\mathbf{p}^i$ for each i. Assume $x \succ z$ and $z \succ y$ at $\mathbf{p}^i$, and consider the following profile.

1. Individual i: $y \succ_i x \succ_i z$
2. Other individuals (denoted by j): $z \succ_j y \succ_j x$

By IIA the social preference is $x \succ z$ and $z \succ y$. Then, by transitivity the social preference about x and y must be $x \succ y$. It means $\neg(y \succ x)$. But by Pareto principle the social preference must be $y \succ x$. Therefore, $x \succ z$ and $z \succ y$ are not consistent at $\mathbf{p}^i$.

10.4 Concluding Remarks

We have examined the Arrow impossibility theorem of social choice theory in an infinite society, and have shown that the theorem that there exists a dictator or there exists no dictator for any social welfare function satisfying Pareto principle and IIA in an infinite society is equivalent to LPO (Limited principle of omniscience), and so it is non-constructive. The assumption of an infinite society seems to be unrealistic. But Mihara (1997) presented an interpretation of an infinite society based on a *finite* number of individuals and a countably infinite number of uncertain states.

10.5 Proof of Lemma 10.1

1. Case 1: There are more than three alternatives.
   Let z and w be alternatives other than x and y, and consider the following profile.
   (a) Individuals in G (denoted by i): $z \succ_i x \succ_i y \succ_i w$.
   (b) Other individuals (denoted by j): $y \succ_j x$, $z \succ_j x$ and $y \succ_j w$. Their preferences about z and w are not specified.

   By Pareto principle the social preference is $z \succ x$ and $y \succ w$. Since G is almost decisive for x against y, the social preference is $x \succ y$. Then, by transitivity the social preference should be $z \succ w$. This means that G is decisive for z against w. From this result we can show that G is decisive for x (or y) against w, for z against x (or y), for y against x, and for x against y. Since z and w are arbitrary, G is decisive about every pair of alternatives, that is, it is a decisive set.
2. Case 2: There are only three alternatives x, y and z.
   Consider the following profile.

(a) Individuals in G (denoted by i): $x \succ_i y \succ_i z$.

(b) Other individuals (denoted by j): $y \succ_j z$, $y \succ_j x$, and their preferences about x and z are not specified.

By Pareto principle the social preference is $y \succ z$. Since G is almost decisive for x against y, the social preference is $x \succ y$. Then, by transitivity the social preference should be $x \succ z$. This means that G is decisive for x against z. Similarly we can show that G is decisive for z against y considering the following profile.

(a) Individuals in G (denoted by i): $z \succ_i x \succ_i y$.

(b) Other individuals (denoted by j): $z \succ_j x$, $y \succ_j x$, and their preferences about y and z are not specified.

By similar procedures we can show that G is decisive for y against z, for z against x, for y against x, and for x against y.

Chapter 11

The Gibbard-Satterthwaite theorem of social choice theory in an infinite society and LPO (Limited principle of omniscience)

This chapter is an attempt to examine the main theorems of social choice theory from the viewpoint of constructive mathematics. We examine the Gibbard-Satterthwaite theorem (Gibbard (1973), Satterthwaite (1975)) in a society with an infinite number of individuals (infinite society). We will show that the theorem that any coalitionally strategy-proof social choice function may have a dictator or has no dictator in an infinite society is equivalent to LPO (Limited principle of omniscience). Therefore, it is non-constructive. A dictator of a social choice function is an individual such that if he strictly prefers an alternative (denoted by x) to another alternative (denoted by y), then the social choice function chooses an alternative other than y. Coalitional strategy-proofness is an extension of the ordinary strategy-proofness. It requires non-manipulability for coalitions of individuals as well as for a single individual[*1].

11.1 Introduction

This chapter is an attempt to examine the main theorems of social choice theory from the viewpoint of constructive mathematics. The Gibbard-Satterthwaite theorem (Gibbard (1973), Satterthwaite (1975)) shows that, with a finite number of individuals, there exists a dictator for any strategy-proof social choice function. In contrast Pazner and Wesley (1977) shows that in an infinite society, there exists a coalitionally strategy-proof social choice function without dictator[*2]. A dictator of a social choice function is an individual such that if he strictly prefers an alternative (denoted by x) to another alternative (denoted by y), then the social choice function chooses an alternative other than y, and it chooses one of his most preferred alternatives. Coalitional strategy-proofness is an extension of the ordinary strategy-proofness. It requires non-manipulability for coalitions of individuals as well as for a single individual.

[*1] This chapter is based on my paper of the same title published in *Applied Mathematics and Computation*, Vol. 193, No. 2, pp. 475-481, 2007, Elsevier.

[*2] Taylor (2005) is a recent book that discusses social choice problems in an infinite society.

In this chapter we will show that the theorem that any coalitionally strategy-proof social choice function may have a dictator or has no dictator in an infinite society is equivalent to LPO (Limited principle of omniscience). Therefore, it is non-constructive.

The omniscience principles are general statements that can be proved classically but not constructively, and are used to show that other statements do not admit constructive proofs*3. This is done by showing that the statement implies the omniscience principle. The strongest omniscience principle is the law of excluded middle. A weaker one is the following limited principle of omniscience (abbreviated as LPO).

Limited principle of omniscience (LPO) Given a binary sequence a_n, $n \in \mathbb{N}$ (the set of positive integers), either $a_n = 0$ for all n or $a_n = 1$ for some n.

In the next section we present the framework of this chapter and some preliminary results. In Section 12.3 we will show the following results.

1. Any coalitionally strategy-proof social choice function may have a dictator or has no dictator, and in the latter case all co-finite sets of individuals (sets of individuals whose complements are finite) are decisive sets (Theorem 11.1).
2. Theorem 11.1 is equivalent to LPO (Theorem 11.2).

A decisive set for a social choice function is a set of individuals such that if individuals in the set prefer an alternative (denoted by x) to another alternative (denoted by y), then the social choice function chooses an alternative other than y regardless of the preferences of other individuals.

11.2 The framework and preliminary results

There are $m(\geq 3)$ alternatives and a countably infinite number of individuals. m is a finite positive integer. The set of individuals is denoted by N. The set of alternatives is denoted by A. N and A are discrete sets*4. For each pair of elements i, j of N we have $i = j$ or $i \neq j$, and for each pair of elements x, y of A we have $x = y$ or $x \neq y$. Each subset of N is a detachable set. Thus, for each individual i of N and each subset I of N we have $i \in I$ or $i \notin I$. The alternatives are represented by x, y, z, w and so on. Denote individual i's preference by $\succ_i$. We denote $x \succ_i y$ when individual i prefers x to y. Individual preferences over the alternatives are transitive weak orders, and they are characterized constructively according to Bridges Bridges (1999). About given three alternatives x, y and z individual i's preference satisfies the following conditions.

1. If $x \succ_i y$, then $\neg(y \succ_i x)$.
2. If $x \succ_i y$, then for each $z \in A$ either $x \succ_i z$ or $z \succ_i y$.

Preference-indifference relation $\succsim_i$ and indifference relation $\sim_i$ are defined by

- $x \succsim_i y$ if and only if $\forall z \in A(y \succ_i z \Rightarrow x \succ_i z)$,
- $x \sim_i y$ if and only if $x \succsim_i y$ and $y \succsim_i x$.

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*3 About omniscience principles we referred to Bridges and Richman (1987), Bridges and Vîţă (2006), Mandelkern (1983) and Mandelkern (1989).

*4 About details of the concepts of discrete set and detachable set, see Bridges and Richman (1987).

Then, the following results are derived.

- $\neg(x \succ_i x)$.
- $x \succ_i y$ entails $x \succsim_i y$.
- The relations $\succ_i$, $\succsim_i$ are transitive, and $x \succsim_i y \succ_i z$ entails $x \succ_i z$.
- $x \succsim_i y$ if and only if $\neg(y \succ_i x)$.

As demonstrated by Bridges (1999) we can not prove constructively that $x \succ_i y$ if and only if $\neg(y \succsim_i x)$.

A combination of individual preferences, which is called a *profile*, is denoted by $\mathbf{p}(= (\succ_1, \succ_2, \cdots))$, $\mathbf{p}'(= (\succ'_1, \succ'_2, \cdots))$ and so on.

We consider social choice functions which choose at least one and at most one alternative corresponding to each profile of the revealed preferences of individuals. We require that social choice functions are *coalitionally strategy-proof*. This means that any group (coalition) of individuals can not benefit by revealing preferences which are different from their true preferences, in other words, each coalition of individuals must have incentives to reveal their true preferences, and they cannot manipulate any social choice function. The coalitional strategy-proofness is an extension of the ordinary strategy-proofness which requires only non-manipulability by an individual. We also require that social choice functions are *onto*, that is, their ranges are A. The Gibbard-Satterthwaite theorem states that, with a finite number of individuals, there exists a dictator for any strategy-proof social choice function, or in other words there exists no social choice function which satisfies strategy-proofness and has no dictator. In contrast Pazner and Wesley (1977) shows that when the number of individuals in the society is infinite, there exists a coalitionally strategy-proof social choice function without dictator. A dictator of a social choice function is an individual one of whose most preferred alternatives is always chosen by the social choice function.

Now we define the following terms.

Decisive If, when all individuals in a group G prefer an alternative x to another alternative y, a social choice function chooses an alternative other than y regardless of the preferences of other individuals, then G is *decisive* for x against y.

Decisive set If a group of individuals is decisive about every pair of alternatives for a social choice function, it is called a decisive set for the social choice function.

The meaning of the term *decisive* is similar to that of the same term used in Sen (1979) for binary social choice rules. G may consist of one individual. If for a social choice function an individual is decisive about every pair of alternatives, then he is a *dictator* of the social choice function.

Further we define the following two terms[*5].

Monotonicity Let x and y be two alternatives. Assume that at a profile $\mathbf{p}$ individuals in a group G prefer x to y, all other individuals (individuals in $N \setminus G$) prefer y to x, and x is chosen by a social choice function. If at another profile $\mathbf{p}'$ individuals in G prefer x to y, then the social choice function chooses an alternative other than y regardless of the preferences of the individuals in $N \setminus G$.

Weak Pareto principle If all individuals prefer x to y, then every social choice function chooses an alternative other than y.

[*5] The concept *monotonicity* is according to Batteau,Blin (and Monjardet). It is different from *strong positive association* by Muller and Satterthwaite (1975) when individual preferences are weak orders (include indifference relations).

We define these terms so as to have constructive nature, and they are slightly different from the definitions in Tanaka (2007b).

We can show the following lemmas.

Lemma 11.1 If a social choice function satisfies coalitional strategy-proofness, then it satisfies monotonicity and weak Pareto principle.

Proof. See Section 11.5. □

Lemma 11.2 Assume that a social choice function is coalitionally strategy-proof. If a group G is decisive for one alternative against another alternative, then it is a decisive set.

Proof. See Section 11.6. □

The implications of this lemma are similar to those of Lemma 3*a in Sen (1979) and Dictator Lemma in Suzumura (2000) for binary social choice rules.

Lemma 11.3 Assume that a social choice function is coalitionally strategy-proof. If two groups G and G' are decisive sets, then their intersection $G \cap G'$ is a decisive set.

Proof. See Section 11.7. □

Note that G and G' can not be disjoint. Assume that G and G' are disjoint. If individuals in G prefer x to y to all other alternatives, and individuals in G' prefer y to x to all other alternatives, then neither G nor G' can be a decisive set. This lemma implies that the intersection of a finite number of decisive sets is also a decisive set.

The proofs of these lemma are almost the same as proofs of Lemma 1, 2, 3 in Tanaka (2007b). But in this chapter we try to present constructive proofs, in particular, the proof of Lemma 11.1.

11.3 Existence of coalitionally strategy-proof social choice function without dictator and LPO

Consider profiles such that one individual (denoted by i) prefers x to y to z to all other alternatives, and all other individuals prefer z to x to y to all other alternatives. Denote such a profile by $\mathbf{p}^i$. By weak Pareto principle any social choice function chooses x or z. If a social choice function chooses x at $\mathbf{p}^i$ for some i, then by monotonicity individual i is decisive for x against z, and by Lemma 11.2 he is a dictator. On the other hand, if a social choice function chooses z at $\mathbf{p}^i$ for all $i \in N$, then there exists no dictator, and a group $N \setminus \{i\}$ is a decisive set for all $i \in N$. By Lemma 11.3 in the latter case all co-finite sets (sets of individuals whose complements are finite sets) are decisive sets. Thus, we obtain the following theorem.

Theorem 11.1 Any coalitionally strategy-proof social choice function may have a dictator or has no dictator, and in the latter case all co-finite sets are decisive sets.

But we can show the following theorem.

Theorem 11.2 Theorem 11.1 is equivalent to LPO.

Proof. We define a binary sequence (a_i) as follows.

$$a_i = 1 \text{ for } i \in \mathbb{N} \text{ if the social choice function chooses } x \text{ at } \mathbf{p}^i$$
$$a_i = 0 \text{ for } i \in \mathbb{N} \text{ if the social choice function chooses } z \text{ at } \mathbf{p}^i$$

The condition of LPO for this binary sequence is as follows.

Limited principle of omniscience (LPO)

$$a_i = 0 \text{ for all } i \in \mathbb{N} \text{ or } a_i = 1 \text{ for some } i \in \mathbb{N}$$

From the arguments before Theorem 11.1 it is clearly equivalent to Theorem 11.1. □

11.4 Concluding Remarks

We have examined the Gibbard-Satterthwaite theorem of social choice theory in an infinite society, and have shown that the theorem that any coalitionally strategy-proof social choice function may have a dictator or has no dictator in an infinite society is equivalent to LPO (Limited principle of omniscience), and so it is non-constructive. The assumption of an infinite society seems to be unrealistic. But Mihara (1997) presented an interpretation of an infinite society based on a *finite* number of individuals and a countably infinite number of uncertain states.

11.5 Proof of Lemma 11.1

We use notations in the definition of monotonicity.

1. (Monotonicity) Let z be an arbitrary alternative other than x and y. Assume that at a profile $\mathbf{p}''$ individuals in G prefer x to y to all other alternatives, and other individuals prefer y to x to all other alternatives. If, when the preferences of some individuals in G change from $\succ_i$ (their preferences at $\mathbf{p}$) to $\succ_i''$ (their preferences at $\mathbf{p}''$), an alternative other than x is chosen by the social choice function, then they can gain benefit by revealing their preferences $\succ_i$ when their true preferences are $\succ_i''$. Thus, the social choice function continues to choose x in this case. By the same logic, when the preferences of all individuals in G change to their preferences at $\mathbf{p}''$, the social choice function chooses x. Next, if, when the preferences of some individuals in $N \setminus G$ change from $\succ_i$ to $\succ_i''$, the social choice function chooses y, then they can gain benefit by revealing their preferences $\succ_i''$ when their true preferences are $\succ_i$. On the other hand, if z is chosen in this case, they can gain benefit by revealing their preferences $\succ_i$ when their true preferences are $\succ_i''$. Thus, x must be chosen. By the same logic, when the preferences of all individuals change to their preferences at $\mathbf{p}''$, the social choice function chooses x. Choice of x by the society never violates the coalitional strategy-proofness.
   Next, if, when the preferences of some individuals in G change from $\succ_i''$ to $\succ_i'$ (their preferences at $\mathbf{p}'$), the alternative chosen by the social choice function changes directly from x to y, then they can gain benefit by revealing their preferences $\succ_i''$ when their true preferences are $\succ_i'$. Thus, the alternative chosen by the social choice function does not directly change from x to y in this case. By the same logic, when the preferences of all individuals in G change to their preferences at $\mathbf{p}'$, the alternative chosen by the social choice function does not directly change from x to y. Further, if, when the preferences of some individuals in $N \setminus G$ change from $\succ_i''$ to $\succ_i'$, the alternative chosen

by the social choice function changes directly from x to y, then they can gain benefit by revealing their preference $\succ_i'$ when their true preferences are $\succ_i''$. By the same logic, when the preferences of all individuals change to their preferences at $\mathbf{p}'$, the alternative chosen by the social choice function does not directly change from x to y.

There is a possibility, however, that the alternative chosen by the social choice function changes from x through $w(\neq x, y)$ to y in transition from $\mathbf{p}''$ to $\mathbf{p}'$. If, when the preferences of some individuals change, the alternative chosen by the social choice function changes from x to w, and further when the preferences of other some individuals (denoted by i) change, the alternative chosen by the social choice function changes to y, they have incentives to reveal their preferences $\succ_i'$ when their true preferences are $\succ_i''$ because they prefer y to w at $\mathbf{p}''$. Therefore, an alternative other than y is chosen by the social choice function at $\mathbf{p}'$. Choice of x or another alternative $w(\neq x, y)$ by the society never violates the coalitional strategy-proofness.

2. (Weak Pareto principle) Let $\mathbf{p}$ be a profile at which all individuals prefer x to y, and $\mathbf{p}'$ be a profile at which x is chosen by the social choice function. Assume that at another profile $\mathbf{p}''$ all individuals prefer x to y to all other alternatives. If, when the preferences of some individuals change from $\succ_i'$ to $\succ_i''$, the social choice function chooses an alternative other than x, then they can gain benefit by revealing their preferences $\succ_i'$ when their true preferences are $\succ_i''$. Thus, x is chosen in this case. By the same logic, when the preferences of all individuals change to their preferences at $\mathbf{p}''$, x is chosen. Since at $\mathbf{p}''$ and at $\mathbf{p}$ all individuals prefer x to y, monotonicity (proved in (1)) implies that an alternative other than y is chosen by the social choice function at $\mathbf{p}$.

   Choice of x or another alternative $w(\neq x, y)$ by the society at $\mathbf{p}$ never violates the coalitional strategy-proofness. For example, let w be an alternative other than x and y and assume that $\mathbf{p}$ is a profile such that all individuals prefer w to x to y to all other alternatives, then w is chosen by any social choice function.

11.6 Proof of Lemma 11.2

1. Case 1: There are more than three alternatives.

   Assume that G is decisive for x against y. Let z and w be given alternatives other than x and y. Consider the following profile.

   (a) Individuals in G prefer z to x to y to w to all other alternatives.

   (b) Other individuals prefer y to w to z to x to all other alternatives.

   By weak Pareto principle the social choice function chooses y or z. Since G is decisive for x against y, z is chosen. Then, by monotonicity the social choice function chooses an alternative other than w so long as the individuals in G prefer z to w. It means that G is decisive for z against w. From this result by similar procedures we can show that G is decisive for x (or y) against w, for z against x (or y), and for y against x. Since z and w are arbitrary, G is decisive about every pair of alternatives, that is, it is a decisive set.

2. Case 2: There are only three alternatives x, y and z.

   Assume that G is decisive for x against y. Consider the following profile.

   (a) Individuals in G prefer x to y to z.

   (b) Other individuals prefer y to z to x.

By weak Pareto principle the social choice function chooses x or y. Since G is decisive for x against y, x is chosen. Then, by monotonicity the social choice function chooses an alternative other than z so long as the individuals in G prefer x to z. It means that G is decisive for x against z. Similarly we can show that G is decisive for z against y considering the following profile.

(a) Individuals in G prefer z to x to y.

(b) Other individuals prefer y to z to x.

By similar procedures we can show that G is decisive for y against z, for z against x, and for y against x.

11.7 Proof of Lemma 11.3

Let x, y and z be given three alternatives, and consider the following profile.

1. Individuals in $G \setminus (G \cap G')$ prefer z to x to y to all other alternatives.
2. Individuals in $G' \setminus (G \cap G')$ prefer y to z to x to all other alternatives.
3. Individuals in $G \cap G'$ prefer x to y to z to all other alternatives.
4. Individuals in $N \setminus (G \cup G')$ prefer z to y to x to all other alternatives.

Since G and G' are decisive sets, the social choice function chooses x. Only individuals in $G \cap G'$ prefer x to z and all other individuals prefer z to x. Thus, by monotonicity $G \cap G'$ is decisive for x against z. By Lemma 11.2 it is a decisive set.

Chapter 12

On the computability of binary social choice rules in an infinite society and the halting problem

This chapter investigates the computability problem of the Arrow impossibility theorem (Arrow (1963)) of social choice theory in a society with an infinite number of individuals (infinite society) according to the computable calculus (or computable analysis) by Aberth (1980) and Aberth (2001). We will show the following results. The problem whether a transitive binary social choice rule satisfying Pareto principle and independence of irrelevant alternatives (IIA) has a dictator or has no dictator in an infinite society is a nonsolvable problem, that is, there exists no ideal computer program for a transitive binary social choice rule satisfying Pareto principle and IIA that decides whether the binary social choice rule has a dictator or has no dictator. And it is equivalent to nonsolvability of the halting problem. A binary social choice rule is a function from profiles of individual preferences to social preferences, and a dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society must also strictly prefer the former to the latter*1.

12.1 Introduction

This chapter investigates the computability problem of the Arrow impossibility theorem (Arrow (1963)) of social choice theory in a society with an infinite number of individuals (infinite society) according to the computable calculus (or computable analysis) by Aberth (1980) and Aberth (2001). Arrow's impossibility theorem shows that, with a finite number of individuals, for any binary social choice rule which satisfies the conditions of transitivity, Pareto principle and independence of irrelevant alternatives (IIA) there exists a dictator. A dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society must also strictly prefer the former to the latter. On the other hand, Fishburn (1970), Hansson (1976) and Kirman and Sondermann (1972) show that, in a society with an infinite number of individuals (infinite society), there exists a transitive binary social choice rule satisfying Pareto principle

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*1 This chapter is based on my paper of the same title which will be published in *Applied Mathematics and Computation*, Elsevier.

and IIA without dictator[*2].

In the next section we present the framework of this chapter and some preliminary results. In Section 13.3 we will show the following results. The problem whether a transitive binary social choice rule satisfying Pareto principle and IIA has a dictator or has no dictator in an infinite society is a nonsolvable problem, that is, there exists no ideal computer program for a transitive binary social choice rule satisfying Pareto principle and IIA that decides whether the binary social choice rule has a dictator or has no dictator. And it is equivalent to nonsolvability of the halting problem.

12.2 The framework and preliminary results

There are more than two (finite or infinite) alternatives and a countably infinite number of individuals. The set of individuals is denoted by ω, and the set of alternatives is denoted by A. The alternatives are represented by x, y, z, w and so on. Individual preferences over the alternatives are transitive linear orders, that is, they prefer one alternative to another alternative, and are not indifferent between them. Denote individual i's preference by $\succ_i$. We denote $x \succ_i y$ when individual i prefers x to y. A combination of individual preferences, which is called a *profile*, is denoted by $\mathbf{p}(= (\succ_1, \succ_2, \cdots))$, $\mathbf{p}'(= (\succ'_1, \succ'_2, \cdots))$ and so on. We assume that the profiles satisfy the free triple property. It means that about any set of three alternatives, the profiles of individual preferences are not restricted. About a set of three alternative (denoted by $\{x, y, z\}$) we denote the set of preferences of individual i by Σ^i_{xyz}. The set of profiles about $\{x, y, z\}$ is denoted by Σ^{ω}_{xyz}, where $\omega = \{1, 2, \cdots\}$ is the set of natural numbers. It represents the set of individuals.

We consider a binary social choice rule about $\{x, y, z\}$ $f : \Sigma^{\omega}_{xyz} \longrightarrow \Sigma_{xyz}$ which determines a social preference about $\{x, y, z\}$ corresponding to each profile. Σ_{xyz} in this formulation denotes the set of social preferences about $\{x, y, z\}$. We denote $x \succ y$ when the society strictly prefers x to y, and denote $x \sim y$ when the society is indifferent between them. The social preference is denoted by $\succ$ at $\mathbf{p}$, by $\succ'$ at $\mathbf{p}'$ and so on.

The social preferences are required to satisfy *transitivity*, *Pareto principle* and *Independence of irrelevant alternatives (IIA)*. The meanings of these conditions are as follows.

Transitivity About three alternatives x, y and z, $x \succ y$ and $y \succ z$ (or $x \succ y$ and $y \sim z$, or $x \sim y$ and $y \succ z$) imply $x \succ z$, and $x \sim y$ and $y \sim z$ imply $x \sim z$.

Pareto principle When all individuals prefer x to y, the society must prefer x to y.

Independence of irrelevant alternatives (IIA) The social preference about every pair of two alternatives x and y is determined by only individual preferences about these alternatives. Individual preferences about other alternatives do not affect the social preference about x and y.

Arrow's impossibility theorem shows that, with a finite number of individuals, for any binary social choice rule which satisfies transitivity, Pareto principle and IIA there exists a dictator. In contrast Fishburn (1970), Hansson (1976) and Kirman and Sondermann (1972) show that when the number of individuals in a society is infinite, there exists a transitive binary social choice rule satisfying Pareto principle and IIA without dictator. A dictator is an individual such that if he strictly prefers an alternative to another alternative, then the society must also strictly prefer the former to the latter.

[*2] Taylor (2005) is a recent book that discusses social choice problems in an infinite society.

■Ideal computer Now we consider an ideal computer according to Aberth (2001). An ideal computer is a machine that manipulates symbol strings, and these symbol strings may be arbitrarily long. The ideal computer may have a finite number of registers. Initially all registers are empty of symbol strings, except for a few registers, $v_1, v_2, \dots, v_n$, this being the inputs to the ideal computer. The outputs of the ideal computer, after it ceases computation, is the contents of another group of registers, $w_1, w_2, \dots, w_m$. If P is the program of the ideal computer, with its registers $v_1, v_2, \dots, v_n$ set to prescribed values a_1, a_2, $\dots$, a_n, respectively, then $P(a_1, a_2, \dots, a_n)$ designates its outputs after computation terminates, that is, the values that leave in $w_1, w_2, \dots, w_m$. An ideal computer for a social choice rule will be explained in the next section.

Next, according to definitions in Sen (1979) we define the following terms.

Almost decisiveness If, when all individuals in a finite or infinite group G prefer an alternative x to another alternative y, and other individuals (individuals in $\omega \setminus G$) prefer y to x, the society prefers x to y ($x \succ y$), then G is *almost decisive* for x against y.

Decisiveness If, when all individuals in a group G prefer x to y, the society prefers x to y regardless of the preferences of other individuals, then G is *decisive* for x against y.

Decisive set If a group of individuals is decisive about every pair of alternatives, it is called a decisive set.

A decisive set may consist of one individual. If an individual is decisive about every pair of alternatives for a binary social choice rule, then he is a *dictator* of the binary social choice rule. Of course, there exists at most one dictator.

First about decisiveness we can show the following lemma.

Lemma 12.1 If a group of individuals G is almost decisive for an alternative x against another alternatives y, then it is decisive about every pair of alternatives, that is, it is a decisive set.

Proof. See Section 12.5. □

The implications of Lemma 12.1 are similar to those of Lemma 3*a in Sen (1979) and Dictator Lemma in Suzumura (2000). Next we show the following lemma.

Lemma 12.2 If G_1 and G_2 are decisive sets, then $G_1 \cap G_2$ is also a decisive set.

Proof. See Section 12.6. □

Note that G_1 and G_2 cannot be disjoint. Assume that G_1 and G_2 are disjoint. If individuals in G_1 prefer x to y, and individuals in G_2 prefer y to x, then neither G_1 nor G_2 can be a decisive set. This lemma implies that the intersection of a finite number of decisive sets is also a decisive set.

These are standard results of social choice theory. But for convenience of readers we present the proofs of these lemmas in the later sections.

12.3 Computability of social choice rules and the halting problem

Consider profiles such that about three alternatives x, y and z one individual (denoted by i) prefers x to y to z, and all other individuals prefer z to x to y. Denote such a profile by $\mathbf{p}^i$, and the set of such profiles is denoted by $\bar{\Sigma}^{\omega}_{x,y,z}$. By Pareto principle the social preference about x and y is $x \succ y$. The social preference is $x \succ z$ or $z \succ y$[*3]. If the social preference is $x \succ z$ at $\mathbf{p}^i$ for some i, then by IIA individual i is almost decisive for x against z, and by Lemma 12.1 he is a dictator. On the other hand if the social preference is $z \succ y$ at $\mathbf{p}^i$ for all $i \in N$, then there exists no dictator. In this case by IIA, Lemma 12.1 and 12.2 all co-finite sets (groups of individuals whose complements are finite sets) are decisive sets. Thus, we obtain

Lemma 12.3 1. Any binary social choice rule which satisfies Pareto principle and IIA has a dictator or has no dictator.

2. In the latter case all co-finite sets are decisive sets.

We can show, however, that for any transitive binary social choice rule satisfying Pareto principle and IIA, the problem whether it has a dictator or has no dictator is a nonsolvable problem, that is, there exists no ideal computer program for a transitive binary social choice rule satisfying Pareto principle and IIA that decides whether it has a dictator or has no dictator.

■Ideal computer for binary social choice rules We consider a program P of an ideal computer for such a transitive binary social choice rule restricted to profiles in $\bar{\Sigma}^{\omega}_{x,y,z}$. The input I of P is a string of individual preferences $(\succ_1, \succ_2, \cdots)$. Possible preferences of each individual about x, y and z and also possible social preferences about x, y and z are, respectively, appropriately enumerated. The ideal computer reads the preference of each individual at the profile $\mathbf{p}^i$, $i = 1, 2, \ldots$, step by step from the preference of individual 1 at $\mathbf{p}^1$, and registers them in sequence in the register v_1. It decides the social preference at $\mathbf{p}^i$, $i = 1, 2, \ldots$, after reading preferences of the first some individuals including individual i, that is, it decides the social preference at $\mathbf{p}^1$ after reading preferences of individuals including individual 1, decides the social preference at $\mathbf{p}^2$ after reading preferences of individuals including individual 2, and so on. And it registers the social preference at each profile in sequence in the register v_2.

If the social preference at $\mathbf{p}^1$ is $x \succ z$, then the ideal computer finds that individual 1 is a dictator, writes "1" in the register w_1 whose value is its output, and it terminates; on the other hand if the social preference at $\mathbf{p}^1$ is $z \succ y$, then the ideal computer does not find a dictator and it continues to read the preference of individual 1 at $\mathbf{p}^2$ in the next step. If the social preference at $\mathbf{p}^2$ is $x \succ z$, then it finds that individual 2 is a dictator, writes "2" in w_1, and it terminates; on the other hand if the social preference at $\mathbf{p}^2$ is $z \succ y$, then it does not find a dictator and it continues to read the preference of individual 1 at $\mathbf{p}^3$ in the next step, and so on. If the binary social choice rule has a dictator, the ideal computer eventually finds a dictator and terminates. On the other hand if the binary social choice rule does not have a dictator, the ideal computer can not find a dictator and it continues computation forever.

We show the following theorem which is the main result of this chapter.

Theorem 12.1 1. For any transitive binary social choice rule satisfying Pareto principle and IIA the problem whether the binary social choice rule has a dictator or has no dictator is a nonsolvable problem, that is, there exists no ideal computer program for any transitive binary social choice rule satisfying Pareto principle and IIA that decides whether it has a dictator or has no dictator.

2. The above result is equivalent to nonsolvability of the halting problem.

[*3] If $x \sim z$ (or $z \succ x$) and $y \sim z$ (or $y \succ z$), transitivity implies $x \sim y$ (or $y \succ x$). It is a contradiction.

Proof. 1. We assume that there is an ideal computer program P^* which solves the problem whether the ideal computer program P for a transitive binary social choice rule finds a dictator or not, that is, it terminates or not. The inputs to the program P^* are a program P in its register v_1 and a string of individual preferences I, which is the input to P, in v_2. P^* analyzes the program P with the input I, and supplies in w_1 a single output integer having two values, 1 to indicate that P finds a dictator, and 0 to indicate that P does not find a dictator. The 0-1 output of P^* is a function of P and I, and then we denote $P^*(P, I)$.

Next we define a program $P'(I)$ such that $P^*(P', I)$ is wrong. First, we construct another program P_S, whose inputs are two programs P^*, P and an integer K. In this formulation K denotes the maximum number of profiles P has read. Thus, we assume that P reads individual preferences until it decides the social preferences at $\mathbf{p}^i$, $i = 1, 2, \dots, K$, or P^* terminates before then. The program $P_S(P^*, P(I), K)$ follows the actions of $P^*(P, I)$ step by step. Then, P_S supplies three output integers. The first output integer is 0 if $P^*(P, I)$ does not terminate after P decides the social preference at $\mathbf{p}^K$, and is 1 if $P^*(P, I)$ terminates just when P decides the social preference at $\mathbf{p}^K$ or before then. If the first output integer is 1, the remaining two output integers are significant, one giving the exact number of K, denoted by K^*, taken by $P^*(P, I)$ to termination, and the other giving the $P^*(P, I)$ output integer, 1 or 0, left in w_1 (of P^*).

The program $P'(I)$ employs P_S as a subroutine and behaves as follows.

(a) If P_S signals termination of $P^*(P', I)$ with the output 1 in w_1 (existence of dictator), then $P'(I)$ gives the result that the social preference about y and z is $z \succ y$ at $\mathbf{p}^i$, $i = 1, 2, \dots$.

(b) If P_S signals termination of $P^*(P', I)$ with the output 0 in w_1 (non-existence of dictator), then $P'(I)$ gives the result that the social preference about x and z is $x \succ z$ at $\mathbf{p}^{K^*}$.

(c) If P_S signals nontermination of $P^*(P', I)$ after P decides the social preference at $\mathbf{p}^K$, then $P'(I)$ gives the result that the social preference about y and z is $z \succ y$ at $\mathbf{p}^i$, $i = 1, 2, \dots, K$.

Thus the binary social choice rule has a dictator or has no dictator, depending on whether P^* claims that it has no dictator or has a dictator, respectively. Whatever result P^* determines for P', the program P^* is wrong. And if P^* never terminate, it is still wrong because it fails to give a valid result that the transitive binary social choice rule has no dictator[*4].

2. According to Aberth (2001) the halting problem is stated as follows.

The halting problem Let P be any program that receives its input I in a single register v_1, and P^* be a program with its inputs P in a register v_1 and I in v_2, and supplies in w_1 a single output integer, 1 to indicate termination for P and 0 to indicate nontermination for P. The halting problem is: Is there a program P^* that can determine whether P with that input will terminate or not terminate?

From the arguments before this theorem and the proof of (1) of this theorem it is clear that nonsolvability of the problem whether any transitive binary social choice rule satisfying Pareto principle and IIA has a dictator or has no dictator is equivalent to nonsolvability of the halting problem.

□

Note: $x \succ z$ and $z \succ y$ are not consistent at $\mathbf{p}^i$ for each i Consider the following profile.

[*4] This proof is based on the proof of nonsolvability of the problem to decide whether any real number equals zero or not in Aberth (2001).

1. Individual i: $y \succ_i x \succ_i z$
2. Other individuals (denoted by j): $z \succ_j y \succ_j x$

Assume $x \succ z$ and $z \succ y$ at $\mathbf{p}^i$ for some i. By IIA the social preference is $x \succ z$ and $z \succ y$. Then, by transitivity the social preference about x and y must be $x \succ y$. But by Pareto principle the social preference must be $y \succ x$. Therefore, $x \succ z$ and $z \succ y$ are not consistent at $\mathbf{p}^i$ for each i.

12.4 Final Remark

We have examined the Arrow impossibility theorem of social choice theory in an infinite society. The assumption of an infinite society seems to be unrealistic. But Mihara (1997) presented an interpretation of an infinite society based on a *finite* number of individuals and a countably infinite number of uncertain states.

12.5 Proof of Lemma 12.1

Consider the following profile.

1. Individuals in G (denoted by i): $x \succ_i y \succ_i z$.
2. Other individuals (denoted by j): $y \succ_j z$, $y \succ_j x$, and their preferences about x and z are not specified.

By Pareto principle the social preference is $y \succ z$. Since G is almost decisive for x against y, the social preference is $x \succ y$. Then, by transitivity the social preference should be $x \succ z$. This means that G is decisive for x against z. Similarly we can show that G is decisive for z against y considering the following profile.

1. Individuals in G (denoted by i): $z \succ_i x \succ_i y$.
2. Other individuals (denoted by j): $z \succ_j x$, $y \succ_j x$, and their preferences about y and z are not specified.

By similar procedures we can show that G is decisive for y against z, for z against x, for y against x, and for x against y.

Interchanging z with another alternative $w \neq x, y, z$, we can show that G is decisive about $\{x, y, w\}$. Similarly we can show that G is decisive about $\{x, v, w\}$, is decisive about $\{u, v, w\}$. u, v and w are arbitrary. Therefore, G is decisive about every pair of alternatives.

12.6 Proof of Lemma 12.2

Consider the following profile about x, y and z.

1. Individuals in $G_1 \setminus G_2$ (denoted by i): $z \succ_i x \succ_i y$
2. Individuals in $G_2 \setminus G_1$ (denoted by j): $y \succ_j z \succ_j x$
3. Individuals in $G_1 \cap G_2$ (denoted by k): $x \succ_k y \succ_k z$
4. Other individuals (denoted by l): $z \succ_l y \succ_l x$

Since G_1 and G_2 are decisive sets, the social preference is $x \succ y$ and $y \succ z$. Then, by transitivity the social preference about x and z should be $x \succ z$. Only individuals in $G_1 \cap G_2$ prefer x to z, and all other individuals prefer z to x. Thus, $G_1 \cap G_2$ is almost decisive for x against z. Then, by Lemma 12.1 it is a decisive set.

Chapter 13

Undecidability of Uzawa equivalence theorem and LLPO (Lesser limited principle of omniscience)

The Uzawa equivalence theorem (Uzawa (1962)) showed (classically) that the existence of Walrasian equilibrium in an economy with continuous excess demand functions is equivalent to Brouwer's fixed point theorem, that is, the existence of a fixed point for any continuous function from an n-dimensional simplex to itself. We examine the Uzawa equivalence theorem from the point of view of constructive mathematics, and show that this theorem, properly speaking, the assumption of the existence of a Walrasian equilibrium price vector in this theorem, implies LLPO (Lesser limited principle of omniscience), and so it is non-constructive*[1].

13.1 Introduction

The existence of Walrasian equilibrium in an economy with continuous excess demand functions is proved by Brouwer's fixed point theorem. It is widely recognized that Brouwer's fixed point theorem is not a constructively provable theorem. The so-called Uzawa equivalence theorem (Uzawa (1962)) showed (classically) that the existence of a Walrasian equilibrium price vector is equivalent to Brouwer's fixed point theorem, that is, the existence of a fixed point for any continuous function from an n-dimensional simplex to itself. However, is this theorem constructively proved? In Velupillai (2006) he said that the Uzawa equivalence theorem implies decidability of the halting problem of the Turing machine. In this chapter we examine the Uzawa equivalence theorem from the point of view of *constructive mathematics*, and show that this theorem, properly speaking, the assumption of the existence of a Walrasian equilibrium price vector in this theorem, implies LLPO (Lesser limited principle of omniscience), and so it is non-constructive.

The omniscience principles are general statements that can be proved classically but not constructively, and are used to show that other statements do not admit constructive proofs*[2]. This is done by showing that the statement implies an omniscience principle. The strongest omniscience principle is the law of

*[1] This chapter is based on my paper of the same title which will be published in *Applied Mathematics and Computation*, Elsevier.

*[2] About omniscience principles we refer to Bridges and Richman (1987), Bridges and Vîţă (2006), Mandelkern (1983) and Mandelkern (1989), .

excluded middle. A weaker one is the following limited principle of omniscience (abbreviated as LPO).

Limited principle of omniscience (LPO) Given a binary sequence $(a_n) = a_n, n \in \mathbb{N}$ (the set of positive integers), then either $a_n = 0$ for all n or $a_n = 1$ for some n.

Another omniscience principle is the following LLPO. It is weaker than LPO.

Lesser limited principle of omniscience (LLPO) Given a binary sequence (a_n) with at most one 1, then either $a_n = 0$ for all even n, or else $a_n = 0$ for all odd n.

In the next section we present the theorem of the existence of Walrasian equilibrium and the Uzawa equivalence theorem with their classical proofs. In Section 14.3 we present some results of constructive mathematics, and prove that the assumption of the existence of Walrasian equilibrium in the Uzawa equivalence theorem implies LLPO.

13.2 Existence of Walrasian equilibrium and the Uzawa equivalence theorem

First we present the theorem of the existence of Walrasian equilibrium in an economy with continuous excess demand functions for the goods and its classical proof. Let Δ be an n-dimensional simplex ($n \geq 2$), and $p = (p_0, p_1, \cdots, p_n)$ be a point on Δ. $p_i \geq 0$ for each i and $\sum_{i=0}^{n} p_i = 1$. The prices of at least two goods are not zero. Thus, $p_i \neq 1$ for all i. Then, the theorem of the existence of Walrasian equilibrium is stated as follows.

Theorem 13.1 (Existence of Walrasian equilibrium) Consider an economy with $n+1$ goods $X_0, X_1, \cdots, X_n$ with a price vector $p = (p_0, p_1, \cdots, p_n)$. Assume that an excess demand function for each good $f_i(p_0, p_1, \cdots, p_n)$, $i = 0, 1, \cdots, n$, is continuous and satisfies the following condition,

$$p_0 f_0 + p_1 f_1 + \cdots + p_n f_n = 0 \text{ (the Walras Law)}.$$

Then, there exists an equilibrium price vector $(p_0^*, p_1^*, \cdots, p_n^*)$ which satisfies $f_i(p_0, p_1, \cdots, p_n) \leq 0$ for all i ($i = 0, 1, \cdots, n$). And when $p_i > 0$ we have $f_i(p_0^*, p_1^*, \cdots, p_n^*) = 0$.

Classical proof. See Section 13.5. □

Next we present the Uzawa equivalence theorem (Uzawa (1962)) which states that the existence of Walrasian equilibrium is equivalent to Brouwer's fixed point theorem, that is, the existence of a fixed point for any continuous function from an n-dimensional simplex to itself, and its classical proof.

Theorem 13.2 (Uzawa equivalence theorem) The existence of Walrasian equilibrium is equivalent to Brouwer's fixed point theorem.

Classical proof. We will show the converse of the previous theorem. Let $\psi = \{\psi_0, \psi_1, \ldots, \psi_n\}$ be an arbitrary continuous function from Δ to Δ, and construct excess demand functions by

$$z_i(p) = \psi_i(p) - p_i \mu(p),\ i = 0, 1, \ldots, n, \tag{13.1}$$

where $p = \{p_0, p_1, \ldots, p_n\}$, and $\mu(p)$ is defined as follows,

$$\mu(p) = \frac{\sum_{i=0}^{n} p_i \psi_i(p)}{\sum_{i=0}^{n} p_i^2}.$$

z_i for $i = 0, 1, \dots, n$ are continuous, and as we will show below, they satisfy the Walras Law. Let multiply p_i to each z_i in (13.1), and summing up them from 0 to n, we obtain

$$\sum_{i=0}^{n} p_i z_i = \sum_{i=0}^{n} p_i \psi_i(p) - \mu(p) \sum_{i=0}^{n} p_i^2 = \sum_{i=0}^{n} p_i \psi_i(p) - \frac{\sum_{i=0}^{n} p_i \psi_i(p)}{\sum_{i=0}^{n} p_i^2} \sum_{i=0}^{n} p_i^2$$
$$= \sum_{i=0}^{n} p_i \psi_i(p) - \sum_{i=0}^{n} p_i \psi_i(p) = 0.$$

Thus, z_i for all i satisfy the conditions of excess demand functions, and by Theorem 13.1 there exists an equilibrium price vector. Let $p^* = \{p_0^*, p_1^*, \dots, p_n^*\}$ be an equilibrium price vector. Then we have

$$\psi_i(p^*) \le \mu(p^*) p_i^*, \tag{13.2}$$

and if $p_i^* \neq 0$, $\psi_i(p^*) = \mu(p^*) p_i^*$. But since $\psi_i(p^*)$ must be non-negative by its definition (a function from Δ to Δ), we have $\psi_i(p^*) = 0$ when $p_i^* = 0$. Therefore, for all i we obtain $\psi_i(p^*) = \mu(p^*) p_i^*$. Summing up them from $i = 0$ to n, we get

$$\sum_{i=0}^{n} \psi_i(p^*) = \mu(p^*) \sum_{i=0}^{n} p_i^*.$$

Because $\sum_{i=0}^{n} \psi_i(p^*) = 1$ and $\sum_{i=0}^{n} p_i^* = 1$, we have $\mu(p^*) = 1$, and so we obtain

$$\psi_i(p^*) = p_i^*, \ i = 0, 1, \dots, n.$$

p^* is a fixed point of ψ. We have shown that any continuous function from Δ to Δ must have a fixed point. □

13.3 Uzawa equivalence theorem and LLPO

13.3.1 Basics of constructive mathematics

About major methods and principal results of constructive mathematics we refer to Bridges and Richman (1987), Bridges and Vîţă (2006), Mandelkern (1983) and Mandelkern (1989). A real number is represented as rational approximations, and is identified with a sequence $x = (x_n)$ of rational numbers that is *regular* in the sense that

$$|x_m - x_n| \le \frac{1}{m} + \frac{1}{n}$$

for all positive integers m and n. Two real numbers x and y are equal if $|x_n - y_n| \le \frac{2}{n}$ for all positive integer n. Some operations on $\mathbb{R}$ (the set of real numbers) are defined as follows:

1. $(x \pm y)_n = x_{2n} \pm y_{2n}$,
2. $|x|_n = |x_n|$

where $(x \pm y)_n$ denotes the n-th term of the real number $x + y$ (or $x - y$), and $|x| = \max(x, -x)$. A real number $x = (x_n)$ is positive ($x > 0$) if there exists n such that $x_n > \frac{1}{n}$, and it is nonnegative ($x \ge 0$) if $x_n > -\frac{1}{n}$ for all n. x is negative ($x < 0$) if $-x$ is positive, that is, there exists n such that $-x_n > \frac{1}{n}$, then $x_n < -\frac{1}{n}$. Similarly, x is nonpositive ($x \le 0$) if $-x$ is nonnegative, that is, $-x_n > -\frac{1}{n}$ for all n, then $x_n < \frac{1}{n}$ for all n. For two real numbers x and y we define $x > y$ to mean $x - y > 0$. We obtain the following properties of positive real numbers.

1. If $x > 0$ and $y > 0$, then $x + y > 0$.
   It is clear.
2. If $x + y > 0$, then $x > 0$ or $y > 0$.
   If $x + y > 0$, there is a positive integer n such that $x_{2n} + y_{2n} > \frac{1}{n} = \frac{1}{2n} + \frac{1}{2n}$. Then, we have $x_{2n} > \frac{1}{2n}$ or $y_{2n} > \frac{1}{2n}$. This means $x > 0$ or $y > 0$.

If $x - y > 0$, for any real number z we have $(x - z) + (z - y) > 0$. Then, $x - z > 0$ or $z - y > 0$.

We need the following results.

Lemma 13.1 1. For any real number x there exists a binary sequence (a_n) such that
   (a) $x \leq 0$ if and only if $a_n = 0$ for all n.
   (b) $x > 0$ if and only if $a_n = 1$ for some n.
   Conversely, for any binary sequence (a_n) there exists a real number satisfying these two conditions. Therefore, for a real number x the property that $x \leq 0$ or $x > 0$ is equivalent to LPO.
2. For any real number x there exists a binary sequence (a_n) with at most one 1 such that
   (a) $x \geq 0$ if and only if $a_n = 0$ for all even n.
   (b) $x \leq 0$ if and only if $a_n = 0$ for all odd n.
   Conversely, for any binary sequence (a_n) with at most one 1 there exists a real number satisfying these two conditions. Therefore, for a real number x the property that $x \leq 0$ or $x \geq 0$ is equivalent to LLPO.

Proof. 1. For each positive integer n we have $x < \frac{1}{n}$ or $x > 0$. Define $a_n = 0$ if $x < \frac{1}{n}$ and $a_n = 1$ if $x > 0$. This defines a binary sequence (a_n). If $a_n = 0$ for all n, we have $x < \frac{1}{n}$ for all n, and it follows that $x \leq 0$. If $x \leq 0$ we have $a_n = 0$ for all n. On the other hand, if $a_n = 1$ for some n, we have $x > 0$. If $x > 0$, there exists an integer n such that $x > \frac{1}{n}$, and then we must have $a_n = 1$ for some n.

Conversely, given a binary sequence (a_n), define

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}.$$

It is clear that $x \leq 0$ if and only if $a_n = 0$ for all n since if $a_n = 1$ for some n, $x \geq \frac{1}{2^n}$. And we have $x > 0$ if and only if $a_n = 1$ for some n since if $a_n = 0$ for all n, we have $x = 0$. Thus, x satisfies two conditions in (1).

2. From (1) of this lemma we can construct a binary sequence (b_n) such that $|x| \leq 0$ if and only if $b_n = 0$ for all n, and $|x| > 0$ if and only if $b_n = 1$ for some n. Construct a binary sequence (a_n) as follows. When $b_1 = 0$, define $a_1 = 0$. When $b_1 = 1$, we have $|x| > 0$, and either $x > 0$ or $x < 0$. If $x > 0$, define $a_1 = 1$ and $a_n = 0$ for all $n \geq 2$. If $x < 0$, define $a_1 = 0$, $a_2 = 1$ and $a_n = 0$ for all $n \geq 3$. Assume $b_1 = 0$. When $b_2 = 0$, define $a_2 = 0$. When $b_2 = 1$, we have either $x > 0$ or $x < 0$. If $x > 0$, define $a_2 = 0$, $a_3 = 1$ and $a_n = 0$ for all $n \geq 4$. If $x < 0$, define $a_2 = 1$ and $a_n = 0$ for all $n \geq 3$. We proceed inductively. If $a_n = 0$ for all even n, $|x| \leq 0$ or $x > 0$, and if $a_n = 0$ for all odd n, $|x| \leq 0$ or $x < 0$. If $|x| \leq 0$, $a_n = 0$ for all n. If $x > 0$, $a_n = 1$ for some odd n and $a_n = 0$ for all even n, and if $x < 0$, $a_n = 1$ for some even n and $a_n = 0$ for all odd n.

Conversely, given a binary sequence (a_n) with at most one 1, define

$$x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} a_n}{2^n}.$$

Then, it is clear that $x \geq 0$ if and only if $a_n = 0$ for all even n since if $a_n = 1$ for some even n, $x = -\frac{1}{2^n}$. Similarly we have $x \leq 0$ if and only if $a_n = 0$ for all odd n since if $a_n = 1$ for some odd n, $x = \frac{1}{2^n}$. Thus, x satisfies two conditions in (2).

□

13.3.2 Uzawa equivalence theorem and LLPO

For all i other than 0 ψ_i is assumed to be defined as follows

$$\psi_i = \frac{\lambda_i(p_i)}{\sum_{j=0}^{n} \lambda_j(p_j)}.$$

And for $i = 0$ we assume

$$\psi_0 = \frac{\lambda_0(p_0)}{\sum_{j=0}^{n} \lambda_j(p_j)}.$$

Then we have

$$z_i(p) = \frac{\lambda_i(p_i)}{\sum_{j=0}^{n} \lambda_j(p_j)} - p_i \frac{\sum_{j=0}^{n} p_j \lambda_j(p_j)}{\sum_{j=0}^{n} p_j^2 \sum_{j=0}^{n} \lambda_j(p_j)}, \text{for all } i \neq 0,$$

and

$$z_0(p) = \frac{\lambda_0(p_0)}{\sum_{j=0}^{n} \lambda_j(p_j)} - p_0 \frac{\sum_{j=0}^{n} p_j \lambda_j(p_j)}{\sum_{j=0}^{n} p_j^2 \sum_{j=0}^{n} \lambda_j(p_j)}.$$

If $z_i = 0$ for all i including $i = 0$, then we obtain

$$p_0 \lambda_i(p_i) = p_i \lambda_0(p_0), \text{for all } i \neq 0. \tag{13.3}$$

Now specifically we assume

$$\lambda_i(p_i) = p_i + 1,\ i \neq 0, \tag{13.4}$$

and

$$\lambda_0(p_0) = \begin{cases} \frac{np_0}{1-p_0} + \frac{1}{4} + b, & \text{when } p_0 < \frac{1}{4} \\ \frac{np_0}{1-p_0} + p_0 + b, & \text{when } \frac{1}{4} \leq p_0 \leq \frac{1}{2} \\ \frac{np_0}{1-p_0} + \frac{1}{2} + b, & \text{when } \frac{1}{2} < p_0 < 1 \end{cases} \tag{13.5}$$

where b is a real number such that $b > -\frac{1}{4}$. From (13.3) and (13.4) we have

$$p_i(\lambda_0(p_0) - p_0) = p_0,\ i \neq 0. \tag{13.6}$$

This implies that all p_i, $i \neq 0$, are equal. Since $\sum_{j=0}^{n} p_j = np_i + p_0 = 1$ we have

$$p_i = \frac{1 - p_0}{n} \tag{13.7}$$

If $p_0 = 0$, we have $p_i = \frac{1}{n}$ for all $i \neq 0$. But, then since $\lambda_0(p_0) = \frac{1}{4} + b > 0$ it contradicts (13.6). Thus $p_0 \neq 0$. From (13.6) and (13.7)

$$(1 - p_0)(\lambda_0(p_0) - p_0) = np_0. \tag{13.8}$$

Therefore, from (13.5) and (13.8) we obtain

$$\begin{cases} p_0 - \frac{1}{4} - b = 0, \text{ when } p_0 < \frac{1}{4} \\ b = 0, \text{ when } \frac{1}{4} \le p_0 \le \frac{1}{2} \\ p_0 - \frac{1}{2} - b = 0, \text{ when } p_0 > \frac{1}{2} \end{cases} \tag{13.9}$$

These are the equilibrium conditions. The assumption of the existence of Walrasian equilibrium implies the existence of p_0 in $(0, 1)$ such that one of these conditions is satisfied. Which of the conditions is satisfied depends on the value of b.

Now we show the following main result of this chapter.

Lemma 13.2 The existence of an equilibrium price vector assumed in the Uzawa equivalence theorem implies LLPO.

Proof. Let p_0^* be an equilibrium value of p_0. If $b < 0$, we have $p_0^* < \frac{1}{4}$. If $b = 0$, p_0^* is any value in $\left[\frac{1}{4}, \frac{1}{2}\right]$. On the other hand, if $b > 0$, we have $p_0^* > \frac{1}{2}$. About three real numbers p_0^*, $\frac{1}{4}$ and $\frac{1}{2}$ we have $p_0^* > \frac{1}{4}$ or $p_0^* < \frac{1}{2}$. If $p_0^* > \frac{1}{4}$, then b must satisfy $b \ge 0$. And if $p_0^* < \frac{1}{2}$, then b must satisfy $b \le 0$. Therefore, in order to determine an equilibrium price p_0^* we must know whether $b \ge 0$ or $b \le 0$. As proved in (2) of Lemma 2 it implies LLPO. □

13.4 Final remark

The Uzawa equivalence theorem in general equilibrium theory demonstrates that the existence of Walrasian equilibrium in an economy with continuous excess demand functions is equivalent to Brouwer's fixed point theorem. We have shown that the existence of equilibrium price vector assumed in the Uzawa equivalence theorem implies LLPO (Lesser limited principle of omniscience). Therefore, it is non-constructive.

13.5 Proof of Theorem 13.1

Let v_i be a function from $p = (p_0, p_1, \cdots, p_n)$ to $v = (v_0, v_1, \cdots, v_n)$ as follows,

$$v_i = p_i + f_i, \text{ when } f_i > 0,$$

$$v_i = p_i, \text{ when } f_i \le 0.$$

We construct a function $\varphi = (\varphi_0, \varphi_1, \cdots, \varphi_n)$ from Δ to Δ as follows.

$$\varphi_i(p_0, p_1, \cdots, p_n) = \frac{1}{v_0 + v_1 + \cdots + v_n} v_i.$$

Since we have $\varphi_i \ge 0,\ i = 0, 1, \cdots, n$, and

$$\varphi_0 + \varphi_1 + \cdots + \varphi_n = 1,$$

$(\varphi_0, \varphi_1, \cdots, \varphi_n)$ is a point on Δ.

Since each f_i is continuous, each φ_i is also continuous. Thus, by Brouwer's fixed point theorem there exists $p^* = (p_0^*, p_1^*, \cdots, p_n^*)$ that satisfies

$$(\varphi_0(p_0^*, p_1^*, \cdots, p_n^*), \varphi_1(p_0^*, p_1^*, \cdots, p_n^*), \cdots, \varphi_n(p_0^*, p_1^*, \cdots, p_n^*)) = (p_0^*, p_1^*, \cdots, p_n^*).$$

Since $v_i \geq p_i$ for all i, we have $v_i(p_0^*, p_1^*, \cdots, p_n^*) = \lambda p_i^*$ for all i for some $\lambda \geq 1$. We will show $\lambda = 1$. Now assume $\lambda > 1$. Then, if $p_i^* > 0$ we have $v_i(p_0^*, p_1^*, \cdots, p_n^*) > p_i^*$, that is, $f_i(p_0^*, p_1^*, \cdots, p_n^*) > 0$. On the other hand, since for all i $p_i^* \geq 0$ and the sum of them is one, at least one of them is positive. Then, we have $p_0^* f_0 + p_1^* f_1 + \cdots + p_n^* f_n > 0$. It contradicts the Walras Law. Therefore, we get $\lambda = 1$. And we obtain $v_0 = p_0^*,\ v_1 = p_1^*, \cdots,\ v_n = p_n^*$ and $f_i(p_0^*, p_1^*, \cdots, p_n^*) \leq 0$ for all i. □

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