# Productive government expenditure and economic growth in a heterogeneous-agents model*

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**Abstract**

This paper examines the relationship between productive government expenditure and economic growth. An R&D-based model of endogenous growth is used in which agents have heterogeneous entrepreneurial abilities. We show that if the entrepreneurial ability follows a *long- and fat-tailed* distribution, then the relationship between government expenditure/GDP and economic growth rate is depicted by an *inverted U-shaped curve with a flat top*. The flat top indicates that government size change has a limited impact on growth. We calibrate the model to U.S. data and empirically confirm the above theoretical prediction.

**Keywords:** endogenous growth, government expenditure, heterogeneous agents, nonlinear relationship

**JEL Classification:** E62, O40

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# 1 Introduction

Does government expenditure size affect long-term growth? What is the optimal government spending level that maximizes growth? Economists have long been discussing these questions through substantial amount of accumulated research. Since the 2008 European sovereign debt crisis, this topic has been the center of debate on fiscal policy.

Barro's (1990) seminal work constructs an endogenous growth model in which government spending is an input in final-good production and taxes are distortionary. Barro demonstrats the possibility of an inverted U-shaped relationship between the size of government expenditure and economic growth. Greater government expenditure means a larger input in the final-good production that incentivizes private investment. However, larger government expenditure also means a higher tax rate, which lowers the net return to private capital, thus reducing private investment. When the size of government expenditure is initially small, the first positive effect dominates the second negative effect. On contrary, when high, the second effect dominates the first one. Barro's (1990) endogenous growth model has been extended and examined, with similar results (Barro and Sala-i-Martin, 1992; Futagami et al., 1992; Turnovsky, 1996; Glomm and Ravikumar, 1997; Fisher and Turnovsky, 1998).

Numerous studies have empirically examined the relationship between government expenditure size and economic growth based on the above theoretical results. However, these empirical studies have not reached a broad consensus. Some authors find a positive relationship between government expenditure size and economic growth,[1] whereas others find a negative relationship.[2] Interestingly, however, a fuzzy relationship is also reported in several studies. For example, Cronovich (1998), Bairam (1990), and Fidrmuc (2003) find a lack of relationship

[1]Sattar (1993) and Bose et al. (2007), for example, find a positive relationship between the size of government and the economic growth rate in developing countries. Using OECD data, Aschauer (1989), Evans and Karras (1994), Kneller et al. (1999), Bleaney et al. (2001), and Colombier (2009) find the same results. Ram (1986) and Easterly and Rebelo (1993) find a positive correlation using data from both developing and developed countries.

[2]Landau (1983, 1986) and Devarajan et al. (1996) find a negative correlation between the size of government expenditure and growth for developing countries. The same results are also found for OECD countries (Ahmed, 1986; Hsieh and Lai, 1994; Fölster and Henrekson, 2001; Afonso and Furceri, 2010; Afonso and Alegre, 2011), and both developing and developed countries (Grier and Tullock, 1989; Landau, 1983, Barro, 1990, 1991). These disparate results comport with the theoretical prediction of either a positive or negative relationship between government size and growth.

between the size of government expenditure and growth in developing countries. The same result is also found for developed countries (Saunders, 1995; Levine and Renelt, 1992; Andrès et al.1997; Ghali, 1999) and for both developed and developing nations (Kormendi and Meguire, 1985; Scully, 1989; Lee and Lin, 1994; Lin, 1994).[3]

Thus, the theoretical challenge is to construct a model that can explain the positive, negative, and fuzzy relationships between government size and economic growth in a single setting. This paper aims to present such an analytical framework to eliminate discrepancies between theoretical and empirical studies.

We construct this analytical framework based on Romer's (1990) endogenous growth model. In this model, R&D activities that increase various intermediate goods drive long-term growth. The final good is produced using a continuum of intermediate goods and productive government spending financed contemporaneously by a flat-rate capital income tax. Our analysis also builds on Jaimovich and Rebelo (2017) and Arawatari et al.'s (2018) approaches. We assume that agents have heterogeneous R&D ability, with an endogenously determined cutoff level. Agents whose abilities are below the cutoff disregard innovation and become workers.

Similar to Barro's (1990) model, a change in government spending has two opposing effects on economic growth in our model. First, high government expenditure increases monopolistic profits and thus stimulates entry of intermediate-good firms, indicating a positive effect on long-term growth rate. Second, high government spending indicates a high tax rate, depressing the net benefit of R&D and indicating a negative effect on long-term growth rate. Therefore, our model generates an inverted U-shaped relationship between government expenditure/GDP ratio (government size) and economic growth, similar to Barro (1990). Moreover, the inverted U-shaped relationship holds regardless of the presence or absence of heterogeneity in ability.

Then, how do heterogeneous abilities affect the relationship between government size and economic growth? Figure 1 illustrates the answer to this question.[4] The solid and dashed lines illustrate the relationship between government expenditure/GDP ratio and economic growth

[3]Facchini and Melki (2013) provide a detailed literature review on empirical studies on the growth effect of government expenditure.

[4]We will discuss this figure in Section 4.

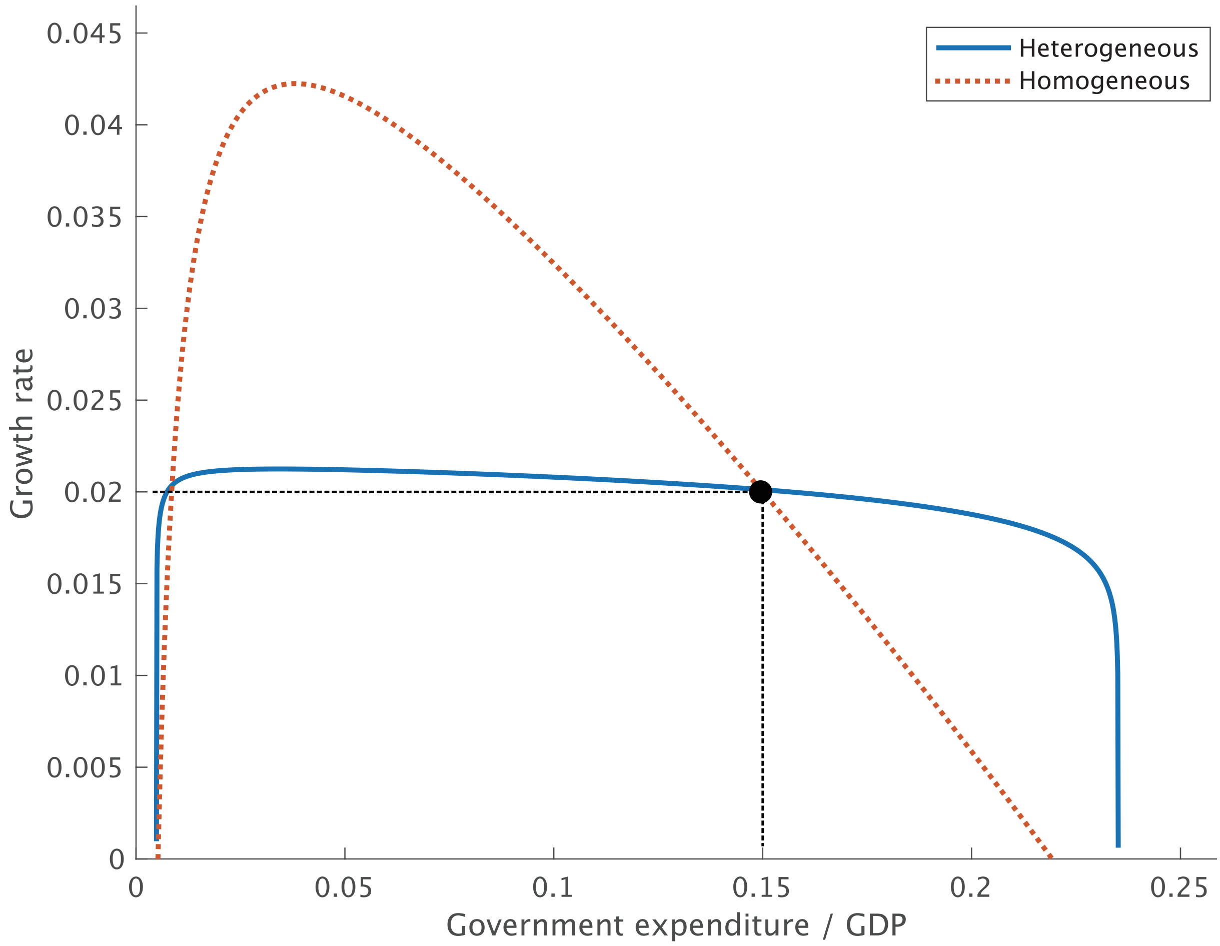


**Figure 1:** The relationship between government size and economic growth rate. The solid line shows the graph of the heterogeneous-ability economy. The dashed line shows the graph of the homogeneous-ability economy.

rate in the heterogeneous- and homogeneous-ability economies, respectively. Both graphs differ numerically, though they show an inverted U-shaped relationship. This inverted U-shaped curve has a *flat top* in the presence of heterogeneity. The flat top illustrates that a change in government size has a limited impact on growth. This result suggests that heterogeneity may be a source of a fuzzy relationship between government size and long-term economic growth. We analytically derive the *inverted U-shaped curve with a flat top* (though Figure 1 shows our calibration results). Thus, heterogeneity's ability to generate a flat top or fuzzy relationship can be easily explained.

If we assume that agents are homogeneous, then government expenditure's positive or negative impact uniformly affects R&D incentives. As a result, government size and economic growth have a normal inverted U-shaped relationship. If agents have heterogeneous R&D ability, then government expenditure size non-uniformly affects their occupational choice. When government expenditure/GDP ratio is sufficiently low or high, R&D's net benefit is small, with only high-ability agents becoming entrepreneurs. Thus, cutoff level changes affect high-ability agents' occupational choice. Given that this impact is relatively large, a change in government size significantly impacts economic growth. In contrast, when government expenditure/GDP ratio is moderate, a change government size generates occupational changes for low-ability agents. Hence, the impact on economic growth is small. Therefore, an *inverted U-shaped curve with a flat top* depicts the relationship between government size and economic growth rate in the case of heterogeneous-ability agents. This flat top explains the positive, negative, and fuzzy relationships between government expenditure/GDP ratio and economic growth rate.

We calibrate the model to U.S. data and empirically confirm our analysis. Assuming that entrepreneurial ability follows a truncated Pareto distribution, our simulation performs an *inverted U-shaped relationship with a flat top* between government expenditure/GDP ratio and economic growth rate under plausible parameter values. A fuzzy relationship exists if government expenditure/GDP ratio is approximately between 2% and 20%. However, long-term growth rate significantly increases or decreases when government expenditure/GDP ratio is outside this range. The U.S. average general government final consumption expenditure (% of GDP) is approximately 15%. Thus, our numerical example suggests that the U.S. economy is on the flat top of the inverted U-shaped curve (see Figure 1). Furthermore, a small government size change is unimportant to the U.S. economy from the economic growth perspective.

Our theoretical and numerical results provide a new perspective of optimal government size. Figure 1 shows that the correlation between government expenditure/GDP ratio and economic growth rate in the heterogeneous-ability economy is not prominent when government expenditure/GDP ratio is moderate. This finding implies that the debate about government size is not significant to economic growth rate unless government size is extremely large or small.

# 2 Model

Time is continuous and is denoted by $t \geq 0$. We consider a Romer's (1990) endogenous growth model in which growth is driven by R&D activities that expand the variety of intermediate goods. Our model differs from Romer (1990) in two ways. First, we assume that government spending is an input of final-good production, as in Barro (1990). Second, following Jaimovich and Rebelo (2017), we assume that agents have heterogeneous entrepreneurial ability. Agents choose whether to become a worker or an entrepreneur, as in Lucas (1978). If an agent becomes an entrepreneur, then he/she engages in R&D activities to increase her intermediate-good firms.

## 2.1 Final-good production

The production technology of the final good is given by the following:

$$Y_t = \left(\frac{G_t}{N_t}\right)^{\theta} \cdot l_t^{\alpha} \cdot \int_0^{N_t} z_{j,t}^{1-\alpha} \mathrm{d}j, \;\; \theta \in (0,1), \quad \alpha \in (0,1), \tag{1}$$

where $Y_t$ is the final-good output, $G_t$ is the productive government spending, $N_t$ is the number of intermediate goods, $l_t$ is labor input, and $z_{i,t}$ is the quantity of intermediate input $j \in [0, N_t]$. Following Barro's (1990) model, total factor productivity depends on government expenditure size, $G_t$. Since $\theta \in (0,1)$, an increase in the number of intermediate goods, $N_t$, under constant government spending, $G_t$, lowers TFP. We can interpret this relationship as a congestion effect. In this model, a larger $N_t$ is associated with a large economy size. Therefore, an increase in $N_t$ under constant $G_t$ lowers TFP due to the congestion associated with the public goods. $\theta$ parameterizes the degree of congestion associated with productive government spending (Turnovsky, 1996).

Assume that the price of final good is normalized to unity. The final-good sector is competitive, and final-good producers maximize after-tax profits as follows:

$$\pi_t^f = (1-\tau_t) \cdot \left\{ \left(\frac{G_t}{N_t}\right)^{\theta} \cdot l_t^{\alpha} \cdot \int_0^{N_t} z_{j,t}^{1-\alpha} \mathrm{d}j - \int_0^{N_t} p_{j,t} z_{j,t} \mathrm{d}j - w_t l_t \right\}, \tag{2}$$

where $\tau_t$ is the capital income tax rate, $p_{j,t}$ is the price of the intermediate good $j$, and $w_t$ is the wage rate expressed in terms of final-good units. The first-order conditions are as follows:

$$\frac{\partial \pi_t^f}{\partial z_{j,t}} = 0 \quad \Leftrightarrow \quad p_{j,t} = (1-\alpha)\left(\frac{G_t}{N_t}\right)^{\theta}\left(\frac{l_t}{z_{j,t}}\right)^{\alpha} \quad \forall j, \tag{3}$$

$$\frac{\partial \pi_t^f}{\partial l_t} = 0 \quad \Leftrightarrow \quad w_t = \alpha\left(\frac{G_t}{N_t}\right)^{\theta}\int_0^{N_t}\left(\frac{l_t}{z_{j,t}}\right)^{\alpha-1} \mathrm{d}j. \tag{4}$$

Equation (3) implies that larger government expenditure shifts the inverse demand curve for intermediate goods upwards. Given that the final-good sector is competitive, the value of $\pi_t^f$ is zero in equilibrium.

## 2.2 Households

Consider a representative "large" household composed of heterogeneous agents. This assumption avoids the complexity involved in managing asset holding distribution. A unit continuum of identical households exists. Thus, the representative "large" household consists of $L$ infinitely lived agents with identical preferences. Following Jaimovich and Rebelo (2017), we assume that agents in the representative household's agents have heterogeneous entrepreneurial ability, $h \in [h_{min}, h_{max}]$. This variable follows a cumulative distribution $F(h)$ that is continuously differentiable. The utility of the representative "large" household at time $s$ is given as follows:

$$U_s = \int_s^{\infty} \frac{(c_t)^{1-\sigma} - 1}{1-\sigma} \cdot e^{-\rho(t-s)}\mathrm{d}t, \tag{5}$$

where $c_t$ is the final-good consumption per agent at time $t$, $\rho > 0$ is the time preference rate, and $\sigma > 0$ is the inverse of inter-temporal substitution.

Each agent in this representative household owns intermediate-good firms. Let $n_{h,t}$ denote the number of intermediate-good firms that an agent with ability $h$ owns. Then, the aggregate number of intermediate-good firms is given by $N_t = \int_{h_{min}}^{h_{max}} n_{h,t} L \mathrm{d}F(h)$. Assume that the intermediate-good sector is monopolistically competitive.

In each period, each agent chooses whether to become a worker in the final-good sector and receive the labor income $w_t$ or become an entrepreneur and receive the monopolistic profits earned by intermediate-good firms. If an agent with ability $h$ becomes an entrepreneur and engages in R&D activities, then he/she can invent $\delta K_t h \mathrm{d}t$ new intermediate goods at time interval $\mathrm{d}t$ and obtain a permanent patent for each. The presence of $K_t$ represents the knowledge spillover (Grossman and Helpman, 1993). The law of motion for $n_{h,t}$ is given by $\dot{n}_{h,t} = \delta K_t h \cdot (1 - I_{h,t})$, where $I_{h,t} = 0$ holds if an agent with ability $h$ becomes an entrepreneur and $I_{h,t} = 1$ holds if he/she becomes a worker. Then, the dynamics of the number of intermediate goods is as follows:

$$\dot{N}_t = \int_{h_{min}}^{h_{max}} \dot{n}_{h,t} L \mathrm{d}F(h) = \delta K_t \cdot \int_{h_{min}}^{h_{max}} h \cdot (1 - I_{h,t}) L \mathrm{d}F(h). \tag{6}$$

Each unit of the intermediate good is produced with $\eta > 0$ units of the final good as variable costs and $\xi > 0$ units of the final good as fixed costs. From Equation (3), the after-tax profit of the intermediate good $j$ is as follows:

$$\pi_{j,t} = (1 - \tau_t) \cdot \left\{ (1 - \alpha) \left( \frac{G_t}{N_t} \right)^{\theta} l_t^{\alpha} z_{j,t}^{1-\alpha} - \eta z_{j,t} - \xi \right\}. \tag{7}$$

Each agent receives a monopolistic profit $\pi_{j,t}$ from the intermediate-good firm that he/she owns. The representative household as a whole receives $\int_0^{N_t} \pi_{j,t} \mathrm{d}j$. The household's flow budget constraint is as follows:

$$Lc_t + \dot{b}_t = r_t b_t + \int_{h_{min}}^{h_{max}} w_t L I_{h,t} \mathrm{d}F(h) + \int_0^{N_t} \pi_{j,t} \mathrm{d}j, \tag{8}$$

where $b_t$ denotes the real bond holdings of the representative household and $r_t$ is the real interest rate. To simplify, we assume that agents with identical ability own the same initial stock of intermediate-good firms and that all agents have zero initial bond holdings.

Given $b_0$, $n_{h,0}$, and $N_0$, the representative household maximizes (5) subject to (6)-(8). Appendix A demonstrates that the usual Euler equation holds as follows:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\sigma} \cdot (r_t - \rho) \quad \forall t \geq 0. \tag{9}$$

Appendix A also demonstrates that all intermediate-good firms produce the same quantity[5] as follows:

$$z_{j,t} = \Psi \left(\frac{G_t}{N_t}\right)^{\frac{\theta}{\alpha}} l_t \equiv z_t, \quad \Psi \equiv \frac{(1-\alpha)^{\frac{2}{\alpha}}}{\eta^{\frac{1}{\alpha}}}. \tag{10}$$

Therefore, all intermediate-good firms have the same value, given by the following:

$$\nu_t = \int_t^{\infty} \pi_s e^{-\int_t^s r_u du} ds. \tag{11}$$

Threshold ability $h_t^*$ makes agents indifferent between being a worker and being an entrepreneur. Agents with ability $h < h_t^*$ become workers in the final-good sector, whereas others become entrepreneurs and engage in R&D activities. Thus, threshold ability $h_t^*$ satisfies the following:

$$w_t = \nu_t \delta K_t h_t^*. \tag{12}$$

The left- and right-hand sides of Equation (12) are the opportunity cost and benefit of being an entrepreneur, respectively. Equations (7) and (11) imply that a change in the capital income tax rate, $\tau_t$, affects $\pi_t$ and $\nu_t$ and hence influences the agents' occupational choice based on the size of tax burden and government expenditure, $G_t$.

In equilibrium, the number of workers (the labor supply for final-good production) is given by $l_t = \{1 - F(h_t^*)\} L$, and the number of entrepreneurs is given by $F(h_t^*)L$. Therefore, Equation (6) can be written as $\dot{N}_t = \delta K_t L \int_{h_t^*}^{h_{max}} h dF(h)$. The following discussion assumes that $K_t = N_t$ (Grossman and Helpman, 1993). Then, the growth rate of $N_t$ is given by the follow-

[5]Since all intermediate-good firms produce the same quantity, hereafter we omit the subscript $i$.

ing:

$$\frac{\dot{N}_t}{N_t} = \delta L \int_{h_t^*}^{h_{max}} h \mathrm{d}F(h) \equiv \phi(h_t^*). \tag{13}$$

High $h_t^*$ indicates few entrepreneurs and less R&D activities. Thus, the growth rate is a decreasing function of $h_t^*$.

## 2.3 Government

Government spending is financed contemporaneously by a flat-rate capital income tax. Given that the final-good sector is competitive, $\pi_t^f = 0$ in equilibrium. Hence, tax revenue is given by $\tau_t N_t \pi_t/(1-\tau_t)$[6]. The government's flow budget constraint at time $t$ is as follows:

$$G_t = \frac{\tau_t}{1-\tau_t} \cdot N_t \pi_t. \tag{14}$$

We assume that the government controls the tax rate to maintain the ratio of government spending to GDP constant over time. Let $g_t \equiv G_t/Y_t$ denote the ratio of government spending to GDP (hereafter referred to as government size), which is constant over time, that is, $g_t = \bar{g} \in [0,1]\ \forall t \geq 0$. Then, Equation (14) is written as follows:

$$\bar{g} = \frac{\tau_t}{1-\tau_t} \cdot \pi_t \cdot \frac{N_t}{Y_t}. \tag{15}$$

## 2.4 Equilibrium dynamics of $h_t^*$

The equilibrium conditions for the asset, final-good, and labor markets are respectively given by the following:

$$b_t = 0. \tag{16}$$

$$Y_t = Lc_t + N_t \eta z_t + N_t \xi + G_t. \tag{17}$$

[6]Note that $\pi_t$ is the after-tax profit of an intermediate-good firm.

$$L = F(h_t^*)L + (1 - F(h_t^*))L = l_t + (1 - F(h_t^*))L. \tag{18}$$

We assume a closed economy. Thus, the net supply of real bonds is zero as in Equation (16).

Pertaining to the steady-state equilibrium, we make the following two assumptions:

**Assumption 1.** $\theta < \min\{\alpha, \sigma\alpha\}$.

**Assumption 2.** $\xi < \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} \bar{g}^{\frac{\theta}{\alpha-\theta}} L^{\frac{\alpha}{\alpha-\theta}} \cdot \{\alpha(1-\alpha) - \bar{g}\}$.

Assumption 1 holds if government spending elasticity with respect to output, $\theta$, is sufficiently small. Assumption 2 holds if intermediate-good production's fixed cost is sufficiently small. Assumptions 1 and 2 ensure the existence of a steady state with positive growth and $\tau_t \in [0, 1]$.

Then, Appendix B demonstrates that the tax rate that satisfies the government budget constraint is given by the following:

$$\tau_t = \frac{\bar{g}}{\alpha(1-\alpha)\left(1 - \frac{\xi}{\Pi(h_t^*; \bar{g})}\right)}. \tag{19}$$

Moreover, $\tau_t \in (0, 1]$ holds when $h_t^* \in [\underline{h}(\bar{g}), h_{max}]$. Appendix B also gives the definitions of $\underline{h}(\bar{g})$ and $\Pi(h_t^*; \bar{g})$. In what follows, we concentrate on the case of $h_t^* \geq \underline{h}(\bar{g})$ in which the government budget is balanced. Appendix B also demonstrates that the equilibrium dynamics of threshold ability $h_t^*$ is given by the following:

$$\text{sign}\frac{\dot{h}_t^*}{h_t^*} = \text{sign}\left\{h_t^* - RHS(h_t^*; \bar{g})\right\}, \tag{20}$$

where

$$RHS(h_t^*; \bar{g}) \equiv \frac{\rho + \sigma\phi(h_t^*)}{\left\{\alpha(1-\alpha)\left(1 - \frac{\xi}{\Pi(h_t^*; \bar{g})}\right) - \bar{g}\right\} \cdot \frac{LF(h_t^*)\delta}{\alpha}}. \tag{21}$$

## 2.5 Steady-state equilibrium

To ensure the existence of a steady state with a positive growth rate, we propose the following assumption:

**Assumption 3.** $\alpha(1-\alpha)\cdot\left\{1-\frac{\xi}{\Pi(h_{max};\bar{g}^*)}\right\}-\bar{g}^*>\frac{\alpha\rho}{L\delta h_{max}}, \quad \bar{g}^*\equiv\frac{1}{\Psi^{1-\alpha}L}\cdot\left(\frac{\theta\xi}{\alpha-\theta}\right)^{\frac{\alpha-\theta}{\alpha}}.$

Assumption 3 holds if the size of representative "large" household, $L$, is large and ensures the existence of a steady state with a positive growth rate.

Define a steady-state equilibrium as an equilibrium where $h_t^*$ is constant over time. Based on Equation (20), the steady-state threshold ability, $h^{*s}$, is characterized by the following:

$$h^{*s}=RHS(h^{*s};\bar{g}). \tag{22}$$

Given that $h^{*s}$ is a function of government size, we express it as $h^{*s}(\bar{g})$ in the following proposition:

**Proposition 1.** *Suppose Assumptions 1-3 hold. A unique steady-state equilibrium with positive economic growth exists, and* $\tau_t\in[0,1]$ *if and only if* $\bar{g}\in[\bar{g}_{min},\bar{g}_{max}]$*, where* $\bar{g}_{max}$ *and* $\bar{g}_{min}$ *satisfy* $RHS(h_{max};\bar{g}_{min})=RHS(h_{max};\bar{g}_{max})=h_{max}$*.*

For the proof, see Appendix C.

When government size is not extreme such that $\bar{g}\in[\bar{g}_{min},\bar{g}_{max}]$, the threshold ability $h^{*s}(\bar{g})$ is smaller than $h_{max}$. Thus, certain agents become entrepreneurs and positive growth is possible.

# 3 Relationship between government size and economic growth

The previous section indicates that a relationship exists between government size, $\bar{g}$, and the growth rate, $\phi(h^{*s}(\bar{g}))$. This section examines the effects of a change in government size on steady-state growth rate. Note that as $F(h)$ is continuously differentiable, $h^{*s}(\bar{g})$ and $\phi(h^{*s}(\bar{g}))$

are also continuously differentiable. Appendix D demonstrates the following:

$$\frac{\mathrm{d}\phi(h^{*s})}{\mathrm{d}\bar{g}} = \frac{\delta L h^{*s} F(h^{*s}) \cdot \left\{ \left( \frac{\bar{g}^*}{\bar{g} F(h^{*s})} \right)^{\frac{\alpha}{\alpha-\theta}} - 1 \right\}}{\frac{F(h^{*s})}{h^{*s} F'(h^{*s})} \cdot \left[ \alpha(1-\alpha) \left( 1 - \frac{\xi}{\Pi(h^{*s};\bar{g})} \right) - \bar{g} \right] + \left[ \alpha\sigma + \alpha(1-\alpha) + \bar{g} \cdot \left\{ \left( \frac{\bar{g}^*}{\bar{g} F(h^{*s})} \right)^{\frac{\alpha}{\alpha-\theta}} - 1 \right\} \right]}. \tag{23}$$

Assumption 2 ensures $\alpha(1-\alpha) - \bar{g} > 0$. Thus, the denominator of the right-hand side of Equation (23) is positive. Therefore, the sign of $\mathrm{d}\phi(h^{*s})/\mathrm{d}\bar{g}$ is the same as the sign of the numerator of the right-hand side of Equation (23). Thus, Lemma 1 can be proved.

**Lemma 1.** *Suppose that Assumptions 1-3 hold. A unique $\bar{g}^{thres} \in (\bar{g}_{min}, \bar{g}_{max})$ exists such that $\bar{g}^* = \bar{g}^{thres} F(h^{*s}(\bar{g}^{thres}))$. Then we obtain the following:*

$$\frac{d\phi(h^{*s}(\bar{g}))}{d\bar{g}} \gtreqless 0 \Leftrightarrow \bar{g} \lesseqgtr \bar{g}^{thres}.$$

For the proof, see Appendix D.

Lemma 1 implies an inverted U-shaped relationship between government size and growth, as in Barro's (1990) model (see Figure 2). A change in the government size has two opposing effects on economic growth. First, government expenditure shifts the inverse demand curve for intermediate goods (see Equation (3)), thereby increasing the monopolistic profits earned by intermediate goods and promoting R&D activities. This positive effect increases the long-term growth rate. Second, larger government spending indicates higher tax rate, which depresses the net benefit of R&D. This negative effect prevents R&D activities and decreases economic growth rate. When government size, $g$, is initially small, the positive effect dominates the negative effect because government spending's marginal productivity is high, and vice versa.

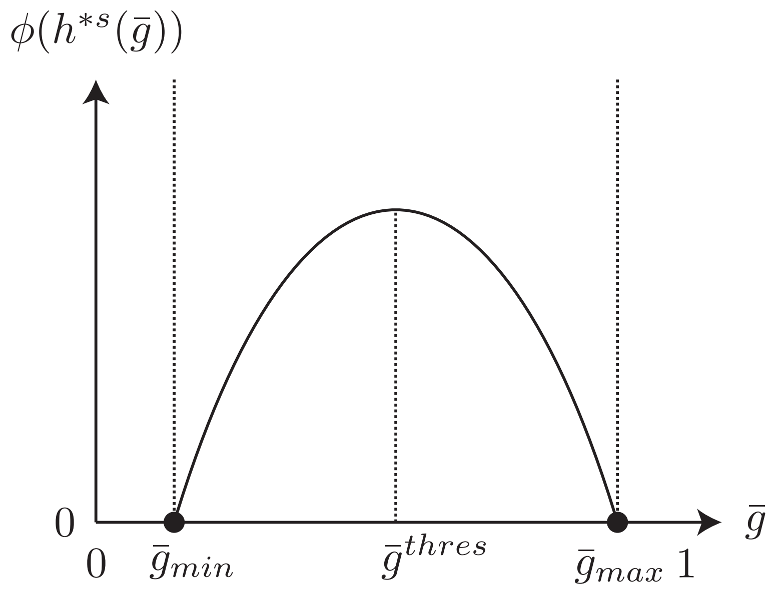


**Figure 2:** Inverted U-shaped relationship between $\bar{g}$ and $\phi(h^{*s}(\bar{g}))$.

## 3.1 Homogeneous-ability economy

Equation (23) indicates that the magnitude of the relationship between government size and economic growth rate depends on the distribution of ability, $F(h)$. We consider a homogeneous-ability economy in which all agents have the same ability $\widehat{h} > 0$ to highlight the role of heterogeneity in entrepreneurial ability.

Hereafter, the variables with superscript $H$ denote those variables for the homogeneous-ability economy. Denote the fraction of workers in the homogeneous-ability economy by $q_t \in [0, 1]$. Then, the growth rate in the homogeneous-ability economy is given by $\phi^H = \delta\widehat{h}L(1-q_t)$.

Denote the steady-state fraction of workers by $q^*$. Then, Appendix E demonstrates that the tax rate that satisfies the government budget constraint is given by the following:

$$\tau_t^H = \frac{\bar{g}}{\alpha(1-\alpha)\left(1 - \dfrac{\xi}{\Pi^H(q_t;\bar{g})}\right)}.$$

Furthermore, $\tau_t^H \in (0, 1]$ holds when $q_t \in [\underline{q}(\bar{g}), 1]$. Appendix E also gives the definition of $\underline{q}(\bar{g})$. In what follows, we concentrate on the case of $q_t \geq \underline{q}(\bar{g})$ in which the government budget is balanced. Appendix E also shows that the equilibrium fraction of workers in the homogeneous-ability economy, $q^*$, is characterized by the following:

$$\widehat{h} = RHS^H(q^*;\bar{g}), \tag{24}$$

where

$$RHS^H(q_t;\bar{g}) \equiv \frac{\rho + \sigma\phi^H(q_t)}{\left\{\alpha(1-\alpha)\left(1 - \frac{\xi}{\Pi^H(q_t;\bar{g})}\right) - \bar{g}\right\} \cdot \frac{Lq_t\delta}{\alpha}}. \tag{25}$$

Pertaining to the steady state, we propose the following assumption:

**Assumption 4.** $\alpha(1-\alpha) \cdot \left\{1 - \frac{\xi}{\Pi^H(1;\bar{g}^*)}\right\} - \bar{g}^* > \frac{\alpha\rho}{L\delta\widehat{h}}$.

The definition of $\bar{g}^*$ is given in Assumption 3, which corresponds to Assumption 4 in the heterogeneous-ability economy. Assumption 4 holds if $L$ is large enough and it ensures the existence of a steady state with positive growth.

Given Assumptions 1, 2, and 4, Proposition 2 below can be proved.

**Proposition 2.** *Suppose that Assumptions 1, 2, and 4 hold. A unique steady-state equilibrium with positive economic growth exists, and* $\tau^H \in [0,1]$ *if and only if* $\bar{g} \in [\bar{g}^H_{min}, \bar{g}^H_{max}]$*, where* $\bar{g}^H_{min}$ *and* $\bar{g}^H_{max}$ *satisfy* $RHS^H(1;\bar{g}^H_{min}) = RHS^H(1;\bar{g}^H_{max}) = \widehat{h}$.

For the proof, see Appendix F.

This proposition corresponds to Proposition 1 in the heterogeneous-ability economy. Then, Appendix G demonstrates the following:

$$\frac{d\phi^H(q^*)}{d\bar{g}} = \frac{\delta L\widehat{h}q^* \cdot \left\{\left(\frac{\bar{g}^*}{\bar{g}q^*}\right)^{\frac{\alpha}{\alpha-\theta}} - 1\right\}}{\alpha\sigma + \alpha(1-\alpha) + \bar{g} \cdot \left\{\left(\frac{\bar{g}^*}{\bar{g}q^*}\right)^{\frac{\alpha}{\alpha-\theta}} - 1\right\}}. \tag{26}$$

Assumption 2 ensures a positive denominator of the right-hand side of Equation (26). Therefore, the sign of $d\phi^H(q^*)/d\bar{g}$ is the same as that of the numerator of the right-hand side of Equation (26). Then, Lemma 2 can be proved as follows:

**Lemma 2.** *Suppose that Assumptions 1, 2, and 4 hold.* $\bar{g}^{thres,H} \in (\bar{g}^H_{min}, \bar{g}^H_{max})$ *such that* $\bar{g}^* = \bar{g}^{thres,H} q^*(\bar{g}^{thres,H})$. *Then we have the following:*

$$\frac{d\phi^H(q^*(\bar{g}))}{d\bar{g}} \gtreqless 0 \Leftrightarrow \bar{g} \lesseqgtr \bar{g}^{thres,H}.$$

For the proof, see Appendix G.

## 3.2 Comparing the heterogeneous- and homogeneous-ability economies

We now compare the heterogeneous- and homogeneous-ability economies to highlight the role of heterogeneity. A comparison of Equations (23) and (26) indicates three differences between these economies: (i) the terms $F(h^{*s})$ and $q^*$ of the two equations, (ii) the terms $h^{*s}$ and $\widehat{h}$ in the numerator of the two equations, and (iii) the first term in the denominator on the right-hand side of Equation (23). The first difference is not fundamental. Both $F(h^{*s})$ and $q^*$ are the steady-state values of the fraction of workers. The second difference shows that in the heterogeneous-ability economy, the impact of government expenditure tends to increase with threshold ability $h^{*s}$. Government expenditure affects economic growth through its impact on agents' occupational choices with the threshold ability. The second difference also suggests that high-ability agents' occupational choices have larger impacts on growth than those of low-ability agents. The third difference is the most important. Using Equation (19), we rewrite the first term in the denominator of Equation (23) as follows:

$$\frac{F(h^{*s})}{h^{*s}F'(h^{*s})} \cdot \left[\alpha(1-\alpha)\left(1 - \frac{\xi}{\Pi(h^{*s};\bar{g})}\right) - \bar{g}\right] = \frac{F(h^{*s})}{h^{*s}F'(h^{*s})} \cdot \frac{(1-\tau)\bar{g}}{\tau}. \tag{27}$$

Therefore, when $\tau \in [0, 1]$, this term is positive. Equation (27) implies that the shape of the distribution function of ability determines the impact of government expenditure on economic growth in the heterogeneous-ability economy. Note that $F'(h^{*s})$ is the density of agents with threshold ability. Therefore, the third difference suggests that when the aggregate size of threshold ability, $h^{*s}F'(h^{*s})$, is large, government expenditure tends to have a large effect on economic growth.

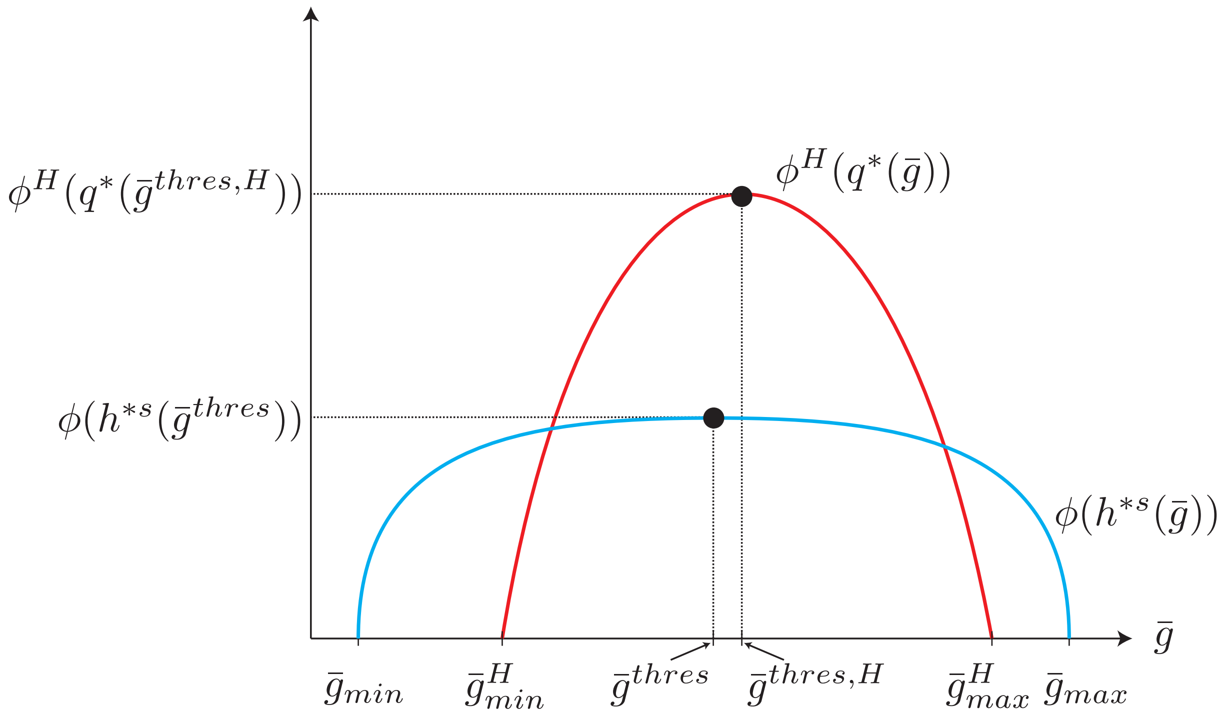


**Figure 3:** The relationship between government size and economic growth rate.

These differences between the homogeneous- and heterogeneous-ability economies produce a different relationship between productive government expenditure and economic growth. Hence, the following proposition is obtained:

**Proposition 3.** *Suppose that $h_{max} > \widehat{h}$ and $\widehat{h}$ is sufficiently large. Then, we obtain the following:*

**(i)** $\bar{g}_{min} < \bar{g}^H_{min} < \bar{g}^H_{max} < \bar{g}_{max}$.

**(ii)** $\max_{\bar{g}} \phi^H(q^*(\bar{g})) > \max_{\bar{g}} \phi(h^{*s}(\bar{g}))$.

For the proof, see Appendix H.

The condition that $h_{max}$ is sufficiently large implies a *long-tailed* distribution of ability. Therefore, Proposition 3 shows that when ability has a wide distribution, the graph of $\phi(h^{*s}(\bar{g}))$ is wider (Proposition 3(i)) and lower (Proposition 3(ii)) than the graph of $\phi^H(q^*(\bar{g}))$ (see Figure 3).

The following proposition also holds.

**Proposition 4.** *Suppose that* $\lim_{h_{max}\to+\infty} h_{max}F'(h_{max}) \neq 0$ *and* $h_{max}$ *is sufficiently large. Then, we have the following:*

$$\begin{aligned}
\frac{d\phi(h^{*s}(\bar{g}))}{d\bar{g}}\Big|_{\bar{g}=\bar{g}_{max}} &< \frac{d\phi^H(q^*(\bar{g}))}{d\bar{g}} \leq 0 \ \forall \bar{g} \in [\bar{g}^{thres,H}, \bar{g}^H_{max}].\\
\frac{d\phi(h^{*s}(\bar{g}))}{d\bar{g}}\Big|_{\bar{g}=\bar{g}_{min}} &> \frac{d\phi^H(q^*(\bar{g}))}{d\bar{g}} \geq 0 \ \forall \bar{g} \in [\bar{g}^H_{min}, \bar{g}^{thres,H}].
\end{aligned}$$

For the proof, see Appendix I.

Proposition 4 suggests that the impact of government expenditure on economic growth in the heterogeneous-ability economy is larger than that in the homogeneous-ability economy when government expenditure size is sufficiently small or large (see Figure 3).

The condition that $h_{max}$ is sufficiently large indicates a *long-tailed* distribution of ability. The condition $\lim_{h_{max}\to+\infty} h_{max}F'(h_{max}) \neq 0$ means that the number of high-ability agents is non-negligible. This relationship implies a *fat-tailed* distribution of ability. A combination the above two conditions leads to a *long- and fat-tailed* distribution of ability.

Propositions 3 and 4 imply that the heterogeneity of entrepreneurial ability plays a key role in the effect of government expenditure on economic growth. In the presence of agents' heterogeneous R&D ability, the linkage between eonomic growth rate and government size is depicted by an *inverted U-shaped curve that has a flat top* for various government spending/GDP ratios. On the one hand, when government expenditure size is sufficiently small or large, it tends to have a large effect on growth (Proposition 4). On the other hand, when it is moderate, the impact of government expenditure size on growth is small (Proposition 3).

The intuition of Propositions 3 and 4 is as follows. Note that when government expenditure is small, the demand for intermediate goods is low (see Equation (3)). Furthermore, when government expenditure is large, the tax rate is high. Therefore, sufficiently small or large government expenditure size leads to a small net profit of an intermediate-good firm. Thus, only high-ability agents become entrepreneurs. Entrepreneurial ability follows a *long- and fat-tailed* distribution. Hence, high-ability agents' occupational choices significantly impact economic growth. This condition results in a strong relationship between government expenditure size and economic growth.

By contrast, when government expenditure is moderate, threshold ability $h_t^*$ is sufficiently low. Note that under the *long- and fat-tailed* distribution of ability, low-ability agents' size is not sufficiently large. Therefore, low-ability agents' occupational choices have less impact on economic growth.

# 4 Quantitative analysis

Section 3 demonstrats that when the entrepreneurial ability follows *long- and fat-tailed* distribution, the relationship between the size of government and the growth rate is depicted by an *inverted U-shaped curve with a flat top.* This section shows that the *flat inverted U-shaped* relationship is obtained under plausible parameter values.

## 4.1 Distribution function

We assume that the entrepreneurial ability follows a truncated Pareto distribution as follows:

$$F(h) = \frac{1-(h_{min}/h)^a}{1-(h_{min}/h_{max})^a}. \tag{28}$$

where $a \geq 1$ is a shape parameter and $h_{min}$ and $h_{max}$ ($h_{max} > h_{min}$) are the lower and upper bounds of ability, respectively. The truncated Pareto distribution is a typical example of a *fat-tailed* distribution. Therefore, when $h_{max}$ is sufficiently large, the above distribution function implies a *long- and fat-tailed* distribution of ability. Thus, the condition in Proposition 4, $\lim_{h_{max}\to+\infty} h_{max}F'(h_{max}) \neq 0$, is satisfied.

## 4.2 Calibration strategy

We now calibrate the model to perform the effects of changes in government size on the growth rate of homogeneous- and heterogeneous-ability economies. Our calibration strategy is based on Jaimovich and Rebelo (2013) and Arawatari et al. (2018).

**Heterogeneous-ability economy**

First, consider the heterogeneous-ability economy whose model features structural parameters $\{\alpha, \theta, \eta, \xi, \sigma, \rho, L, h_{min}, h_{max}, a, \delta\}$ and the policy instrument $\bar{g} \in [0, 1]$. We set the labor share in the final-good production to 60% ($\alpha = 0.6$). We assume that $\sigma = 2$ and $\rho = 0.01$, which are conventional values. The size of the representative "large" household is normalized to 1 ($L = 1$). We normalize the marginal cost of intermediate-good production to unity, ($\eta = 1$), and its fixed cost is not extremely large ($\xi = 0.0001$). Without loss of generality, we set the lower bound of ability to 1, $h_{min} = 1$. Finally, we assume that $\theta = 0.3$.

Three conditions are needed to pin down the values of the remaining three parameters: the upper bound of ability ($h_{max}$), the shape parameter of the truncated Pareto distribution ($a$), and the strength of knowledge spillover, $\delta$. We calibrate the values of these parameters based on the following three empirical facts of the U.S. economy:

1. Average annual GDP per capita growth rate is about 2%, and the ratio of the average general government final consumption expenditure (% of GDP) is approximately 15%.

2. According to the U.S. Census Bureau's "2017 SUSB Annual Data Tables by Establishment Industry," the largest 1,100 U.S. firms with more than 10,000 employees employ 37,739,206 workers in 2017.[7] The U.S. had 5,996,900 firms and 128,591,812 workers, indicating that the top 0.018% of U.S. firms contributed 29.348% of total employment in 2017.

3. Similarly, the top 0.336% of U.S. firms contributed 52.908% of total employment in 2017.

Before calibrating $\{h_{max}, a, \delta\}$, consider the relationship between the size of intermediate-good firms and entrepreneurs' ability. Following Jaimovich and Rebelo (2013), we assume that the intermediate-good and the final-good sectors are vertically integrated. Thus, that intermediate-good firms hire workers to produce the final good. Now, let $s_{h,t} \equiv n_{h,t}/N_t$, denote

[7] Source: https://www.census.gov/data/tables/2017/econ/susb/2017-susb-annual.html. (Accessed on April 14, 2020)

the share of intermediate-good firms owned by an agent with ability $h$, and assume the following:

$$s_{h,0} = \frac{n_{h,0}}{N_0} = \frac{h}{L\int_{h^*}^{h_{max}} hdF(h)}. \tag{29}$$

This assumption implies that intermediate-good firms' initial ownership is distributed among entrepreneurs in proportion to their ability. Under this assumption, we obtain $\dot{n}_{h,t}/n_{h,t} = \phi\left(h^{*s}(\bar{g})\right)\ \forall h \in [h^{*,s}, h_{max}]$. Hence, the share of intermediate-good firms owned by agents, $s_{h,t}$, remains constant over time.[8] Then, recall that intermediate-good firms are symmetric and produce the same quantity. Thus, the number of intermediate goods owned by an agent is proportional to the number of workers he/she employs. Therefore, the total employment is proportional to the entrepreneur's ability.

We calibrate $\{h_{max}, a, \delta\}$ as follows. Total employment is proportional to the entrepreneur's ability. Thus, the abovementioned three empirical facts of the U.S. economy are written as follows:

1. $\phi\big(h^{*s}(\bar{g} = 0.15)\big) = \delta LH\big(h^{*s}(\bar{g} = 0.15)\big) = 0.02.$
2. $\dfrac{\int_{h_1}^{h_{max}} h\mathrm{d}F(h)}{\int_{h^{*s}}^{h_{max}} h\mathrm{d}F(h)} = 0.29348$, where $h_1$ satisfies $\dfrac{\int_{h_1}^{h_{max}} \mathrm{d}F(h)}{\int_{h^{*s}}^{h_{max}} \mathrm{d}F(h)} = 0.00018.$
3. $\dfrac{\int_{h_2}^{h_{max}} h\mathrm{d}F(h)}{\int_{h^{*s}}^{h_{max}} h\mathrm{d}F(h)} = 0.52908$, where $h_2$ satisfies $\dfrac{\int_{h_2}^{h_{max}} \mathrm{d}F(h)}{\int_{h^{*s}}^{h_{max}} \mathrm{d}F(h)} = 0.00336.$

Using an iterative process, we compute $\{h_{max}, a, \delta\}$ to satisfy the above equations. Then, we obtain $a = 1$, $\delta = 0.0016$, and $h_{max} = 46,141,337$.[9] The calibrated value of $h_{max}$ seems sufficiently large. Hence, ability has a *long-tailed* distribution.

[8] From (13) and (29), we obtain

$$\frac{\dot{n}_{h,0}}{n_{h,0}} = \frac{\delta h}{s_{h,0}} = \delta L \int_{h^{*s}}^{h_{max}} h\mathrm{d}F(h) = \phi\left(h^{*s}(\bar{g})\right)\ \forall h \in [h^{*s}, h_{max}]\,.$$

This implies that $n_{h,t}$ grows at the same rate as $N_t$, and the share of patents, $s_{h,t}$, remains constant over time.

[9] Under these parameters and $\bar{g} = 0.15$, our model indicates that the top 0.018% of firms employ 29.3% of workers, and the top 0.336% of firms employ 53.2% of workers, which fit well with the U.S. data.

#### Homogeneous-ability economy

Next, we consider the homogeneous-ability economy whose model features structural parameters $\{\alpha, \theta, \eta, \xi, \sigma, \rho, L, \delta, \widehat{h}\}$ and policy instrument $\bar{g} \in [0,1]$. We use the same parameter values as those in the heterogeneous-ability economy, except for $\widehat{h}$. Given these parameters, we choose $\widehat{h}$. Thus, the steady-state growth rate under $\bar{g} = 0.15$ is 2%, that is, $\phi^H\big(q^*(\bar{g} = 0.15)\big) = \delta\widehat{h}L\big(1 - q^*(\bar{g} = 0.15)\big) = 0.02$. This condition yields $\widehat{h} = 236$, satisfying Proposition 3, $h_{max} > \widehat{h}$.

### 4.3 Results

Figure 1 shows the effect of changes in government size, $\bar{g}$, on the growth rate of both heterogeneous- and homogeneous-ability economies. Both economies have an inverted U-shaped relationship between government size and economic growth rate, similar to Barro (1990). However, the magnitude of these relationships significantly differs when government size is not extremely large or small. In the homogeneous-ability economy, government size strongly impacts growth. As government expenditure/GDP ratio increases from 5% to 20%, economic growth rate decreases from 4.14% to 0.59%.

On the contrary, the effect on growth in the heterogeneous-ability economy is significantly weak. We find an *inverted U-shaped relationship with a flat top* between government size and economic growth. Thus, as government expenditure/GDP ratio increases from 5% to 20%, economic growth rate decreases from 2.12% to 1.88%. This reduction is much smaller than that implied by the model of the homogeneous-ability economy model.

This result agrees with Propositions 3 and 4. Heterogeneity in ability generates the *inverted U-shaped relationship with a flat top* between government size and economic growth. This result suggests a new insight into the magnitude of the effect of government size on growth, though this relationship is consistent with Barro's (1990) findings.

## 4.4 Implications

Barro (1990) shows in his seminal paper that economic growth (and welfare) maximizing level of the government expenditure/GPD ratio is equal to labor share using a simple AK-style endogenous growth model with productive government expenditure. This result implies an optimal government expenditure/GPD of approximately 0.7, and most countries' government size is below the optimal level. Subsequent theoretical studies have modified Barro's (1990) model in several ways and found that the optimal government expenditure/GPD ratio is much lower than that induced by Barro (1990). Based on these theoretical results, several empirical studies have estimated the optimal government size in each country. The debate about the optimal government size has been one of the big issues in economics.

The present study provides a new perspective on the optimal size of government. Figure 1 shows that the correlation between government expenditure/GDP ratio and economic growth rate in the heterogeneous-ability economy is not prominent when the government expenditure/GDP ratio is moderate. This finding implies that the debate about government size is unimportant to economic growth rate unless government size is extremely large or small. For example, our numerical results in Figure 1 suggests that the U.S. economy is on the flat top of the inverted U-shaped curve, and a slight change in government size is unimportant to the U.S. economy from the economic growth perspective.

# 5 Conclusion

This study presents an analytical framework that can explain the positive, negative, and fuzzy relationships between government size and economic growth in a single setting. Using an R&D-based endogenous growth model, we show that a *long- and fat-tailed* distribution of entrepreneurial ability plays a key role in generating an *inverted U-shaped relationship with a flat top* between government expenditure/GDP ratio and economic growth rate. We also calibrate the model to U.S. data and show an *inverted U-shaped relationship with a flat top* between government expenditure/GDP ratio and economic growth rate under plausible parameter values. Our theoretical and numerical results suggest that the debate about government size is unimportant to economic growth rate unless government size is extremely large or small.

# Appendix

## A Utility maximization of households

The first-order conditions of the utility maximization of representative household are given by

$$
\begin{aligned}
c_t &: \ c_t^{-\sigma} = \lambda_t L, && \text{(A.1)}\\
z_{j,t} &: \ (1-\alpha)^2 \left(\frac{G_t}{N_t}\right)^{\theta} l_t^{\alpha} z_{j,t}^{-\alpha} = \eta, && \text{(A.2)}\\
b_t &: \ \dot{\lambda}_t = (\rho - r_t)\lambda_t, && \text{(A.3)}\\
N_t &: \ \lambda_t \cdot (1-\tau_t) \cdot \left\{ (1-\alpha)\left(\frac{G_t}{N_t}\right)^{\theta} l_t^{\alpha} z_{N_t,t}^{1-\alpha} - \eta z_{N_t,t} - \xi \right\} = -\dot{\zeta}_t + \rho\zeta_t, && \text{(A.4)}\\
I_{h,t} &: \ I_{h,t} = \begin{cases} 1 & \text{if } \lambda_t w_t > \zeta_t \delta K_t h \\ 0 & \text{if } \lambda_t w_t \le \zeta_t \delta K_t h \end{cases} && \text{(A.5)}
\end{aligned}
$$

where $\lambda_t$ and $\zeta_t$ are the co-state variables associated with the budget constraint and the law of motion for $N_t$, respectively.

From (A.2), we know that all intermediate-good firms produce the same quantity, as shown in equation (10). Since all intermediate-good firms are symmetric, we can eliminate the subscript $j$ from $z_{j,t}$ in what follows. Substituting (10) into (A.4), we obtain

$$
\lambda_t \underbrace{(1-\tau_t) \cdot \left\{ (1-\alpha)\left(\frac{G_t}{N_t}\right)^{\theta} l_t^{\alpha} z_t^{1-\alpha} - \eta z_t - \xi \right\}}_{\pi_t} = -\dot{\zeta}_t + \rho\zeta_t. \tag{A.6}
$$

Let us define $\nu_t \equiv \zeta_t/\lambda_t$. Substituting (A.3) into the above equation, we obtain $r_t\nu_t = \dot{\nu}_t + \pi_t$, which has the following solution:

$$
\nu_t = \int_t^{\infty} \pi_s e^{-\int_t^s r_u du} ds. \tag{A.7}
$$

Therefore, $\nu_t$ represents the value of an intermediate-good firm.

In equilibrium, the threshold ability $h_t^*$ makes agents indifferent between being a worker and being an entrepreneur. Thus, from equation (A.5), $h_t^*$ satisfies $w_t = \nu_t \delta K_t h_t^*$. From (A.2), all intermediate-good firms produce the same quantity:

$$z_{j,t} = \Psi \left( \frac{G_t}{N_t} \right)^{\frac{\theta}{\alpha}} l_t \equiv z_t, \quad \Psi \equiv \frac{(1-\alpha)^{\frac{2}{\alpha}}}{\eta^{\frac{1}{\alpha}}}.$$

■

# B Equilibrium dynamics

## B.1 The Euler equation

Inserting (10) and (18) into (1) yields

$$\begin{aligned} Y_t &= \left( \frac{G_t}{N_t} \right)^{\theta} l_t^{\alpha} N_t z_t^{1-\alpha} \\ &= \left( \frac{G_t}{N_t} \right)^{\theta} l_t^{\alpha} N_t \Psi^{1-\alpha} \left( \frac{G_t}{N_t} \right)^{\frac{\theta(1-\alpha)}{\alpha}} l_t^{1-\alpha} \\ &= N_t \Psi^{1-\alpha} \left( \frac{G_t}{N_t} \right)^{\frac{\theta}{\alpha}} L F(h_t^*). \end{aligned} \tag{B.1}$$

Using (B.1), we rewrite into the equilibrium condition for the final-good market, (17), as

$$N_t \Psi^{1-\alpha} \left( \frac{G_t}{N_t} \right)^{\frac{\theta}{\alpha}} L F(h_t^*) = L c_t + N_t \eta \Psi \left( \frac{G_t}{N_t} \right)^{\frac{\theta}{\alpha}} L F(h_t^*) + N_t \xi + G_t. \tag{B.2}$$

We define $\widehat{c}_t \equiv c_t / N_t$. Differentiating this equation with respect to time and inserting (9) and (13) into it yields

$$\frac{\dot{\widehat{c}}_t}{\widehat{c}_t} = \frac{1}{\sigma} \cdot (r_t - \rho) - \phi(h_t^*). \tag{B.3}$$

Dividing both sides of (B.2) by $N_t$, we obtain

$$\widehat{c}_t = \left( \frac{G_t}{N_t} \right)^{\frac{\theta}{\alpha}} F(h_t^*) \cdot \left( \Psi^{1-\alpha} - \eta \Psi \right) - \frac{\xi}{L} - \frac{1}{L} \cdot \frac{G_t}{N_t}. \tag{B.4}$$

Inserting the definition of $\bar{g}$, $G_t = \bar{g}Y_t$, into (B.1) yields

$$
\begin{aligned}
Y_t &= (\bar{g}Y_t)^{\frac{\theta}{\alpha}} N_t^{\frac{\alpha-\theta}{\alpha}} \Psi^{1-\alpha} LF(h_t^*) \\
&\Leftrightarrow \quad Y_t^{\frac{\alpha-\theta}{\alpha}} = \bar{g}^{\frac{\theta}{\alpha}} N_t^{\frac{\alpha-\theta}{\alpha}} \Psi^{1-\alpha} LF(h_t^*) \\
&\Leftrightarrow \quad \frac{Y_t}{N_t} = \bar{g}^{\frac{\theta}{\alpha-\theta}} \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} \left\{LF(h_t^*)\right\}^{\frac{\alpha}{\alpha-\theta}} . \qquad \text{(B.5)}
\end{aligned}
$$

Note that Assumption 1 ensures that the final-good production is an increasing function of the size of government, $\bar{g}$, and the labor input, $LF(h_t^*)$.

Using (B.5), we obtain

$$
\frac{G_t}{N_t} = \frac{\bar{g}Y_t}{N_t} = \bar{g}^{\frac{\alpha}{\alpha-\theta}} \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} L^{\frac{\alpha}{\alpha-\theta}} F(h_t^*)^{\frac{\alpha}{\alpha-\theta}}. \qquad \text{(B.6)}
$$

Inserting (B.6) into (B.4), we rewrite the equilibrium condition for the final-good market as

$$
\begin{aligned}
\widehat{c}_t &= (\Psi^{1-\alpha} - \eta\Psi) F(h_t^*) \bar{g}^{\frac{\theta}{\alpha-\theta}} \Psi^{\frac{\theta(1-\alpha)}{\alpha-\theta}} L^{\frac{\theta}{\alpha-\theta}} F(h_t^*)^{\frac{\theta}{\alpha-\theta}} \\
&\quad -\frac{\xi}{L} - \frac{1}{L} \cdot \bar{g}^{\frac{\alpha}{\alpha-\theta}} \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} L^{\frac{\alpha}{\alpha-\theta}} F(h_t^*)^{\frac{\alpha}{\alpha-\theta}} \\
&= \left[(\Psi^{1-\alpha} - \eta\Psi)\bar{g}^{\frac{\theta}{\alpha-\theta}} \Psi^{\frac{\theta(1-\alpha)}{\alpha-\theta}} - \bar{g}^{\frac{\alpha}{\alpha-\theta}} \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}}\right] L^{\frac{\theta}{\alpha-\theta}} F(h_t^*)^{\frac{\alpha}{\alpha-\theta}} - \frac{\xi}{L} \\
&= \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} \left[(\Psi^{1-\alpha} - \eta\Psi)\bar{g}^{\frac{\theta}{\alpha-\theta}} \Psi^{\frac{\theta(1-\alpha)-\alpha(1-\alpha)}{\alpha-\theta}} - \bar{g}^{\frac{\alpha}{\alpha-\theta}}\right] L^{\frac{\theta}{\alpha-\theta}} F(h_t^*)^{\frac{\alpha}{\alpha-\theta}} - \frac{\xi}{L} \\
&= \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} \left[(\Psi^{1-\alpha} - \eta\Psi)\bar{g}^{\frac{\theta}{\alpha-\theta}} \Psi^{-(1-\alpha)} - \bar{g}^{\frac{\alpha}{\alpha-\theta}}\right] L^{\frac{\theta}{\alpha-\theta}} F(h_t^*)^{\frac{\alpha}{\alpha-\theta}} - \frac{\xi}{L} \\
&= \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} \left[(1 - \eta\Psi^{\alpha})\bar{g}^{\frac{\theta}{\alpha-\theta}} - \bar{g}^{\frac{\alpha}{\alpha-\theta}}\right] L^{\frac{\theta}{\alpha-\theta}} F(h_t^*)^{\frac{\alpha}{\alpha-\theta}} - \frac{\xi}{L} \\
&= \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} \bar{g}^{\frac{\theta}{\alpha-\theta}} \left[\alpha(2-\alpha) - \bar{g}\right] L^{\frac{\theta}{\alpha-\theta}} F(h_t^*)^{\frac{\alpha}{\alpha-\theta}} - \frac{\xi}{L}. \qquad \text{(B.7)}
\end{aligned}
$$

Differentiating (B.7) with respect to time yields

$$
\begin{aligned}
\dot{\widehat{c}}_t &= \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} \bar{g}^{\frac{\theta}{\alpha-\theta}} \left[\alpha(2-\alpha) - \bar{g}\right] L^{\frac{\theta}{\alpha-\theta}} \cdot \frac{\alpha}{\alpha-\theta} \cdot F(h_t^*)^{\frac{\theta}{\alpha-\theta}} F'(h_t^*) h_t^* \cdot \frac{\dot{h}_t^*}{h_t^*} \\
&= \frac{\alpha\Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} L^{\frac{\theta}{\alpha-\theta}} F(h_t^*)^{\frac{\alpha}{\alpha-\theta}} \bar{g}^{\frac{\theta}{\alpha-\theta}}}{\alpha-\theta} \cdot \left\{\alpha(2-\alpha) - \bar{g}\right\} \cdot \frac{F'(h_t^*)h_t^*}{F(h_t^*)} \cdot \frac{\dot{h}_t^*}{h_t^*}. \qquad \text{(B.8)}
\end{aligned}
$$

Inserting (B.8) into (B.3), we rewrite the Euler equation as

$$\begin{aligned}&\frac{1}{\sigma}\cdot(r_t-\rho)-\phi(h_t^*)\\&\quad=\frac{1}{\widehat{c}_t}\cdot\frac{\alpha\Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}}L^{\frac{\theta}{\alpha-\theta}}F(h_t^*)^{\frac{\alpha}{\alpha-\theta}}\bar{g}^{\frac{\theta}{\alpha-\theta}}}{\alpha-\theta}\cdot\{\alpha(2-\alpha)-\bar{g}\}\cdot\frac{F'(h_t^*)h_t^*}{F(h_t^*)}\cdot\frac{\dot{h}_t^*}{h_t^*}.\end{aligned}\tag{B.9}$$

## B.2 Occupational choice

Inserting (10) and (B.6) into (12), we obtain

$$\begin{aligned}w_t &= \alpha\left(\frac{G_t}{N_t}\right)^{\theta}N_t l_t^{\alpha-1}\Psi^{1-\alpha}\left(\frac{G_t}{N_t}\right)^{\frac{\theta(1-\alpha)}{\alpha}}l_t^{1-\alpha}\\&= \alpha\left(\frac{G_t}{N_t}\right)^{\frac{\theta}{\alpha}}N_t\Psi^{1-\alpha}\\&= \alpha\bar{g}^{\frac{\theta}{\alpha-\theta}}\Psi^{\frac{\theta(1-\alpha)}{\alpha-\theta}}L^{\frac{\theta}{\alpha-\theta}}F(h_t^*)^{\frac{\theta}{\alpha-\theta}}N_t\Psi^{1-\alpha}\\&= \alpha\Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}}\bar{g}^{\frac{\theta}{\alpha-\theta}}L^{\frac{\theta}{\alpha-\theta}}F(h_t^*)^{\frac{\theta}{\alpha-\theta}}N_t.\end{aligned}\tag{B.10}$$

Recall that we assumed $K_t = N_t$. Using (B.10), we can rewrite equation (12) as

$$\alpha\Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}}\bar{g}^{\frac{\theta}{\alpha-\theta}}L^{\frac{\theta}{\alpha-\theta}}F(h_t^*)^{\frac{\theta}{\alpha-\theta}}=\nu_t\delta h_t^*.\tag{B.11}$$

Taking logarithms and differentiating both sides of (B.11) yields

$$\frac{\theta}{\alpha-\theta}\cdot\frac{F'(h_t^*)h_t^*}{F(h_t^*)}\cdot\frac{\dot{h}_t^*}{h_t^*}=\frac{\dot{\nu}_t}{\nu_t}+\frac{\dot{h}_t^*}{h_t^*}.\tag{B.12}$$

Using the definition of $\nu_t$, (A.3) and (A.6), we obtain

$$\begin{aligned}\frac{\dot{\nu}_t}{\nu_t} &= \frac{\dot{\zeta}_t}{\zeta_t}-\frac{\dot{\lambda}_t}{\lambda_t}\\&= \rho-\frac{\lambda_t}{\zeta_t}\cdot\pi_t-(\rho-r_t)\\&= r_t-\frac{\pi_t}{\nu_t}.\end{aligned}\tag{B.13}$$

Inserting (10) and (B.6) into (7) yields

$$
\begin{aligned}
\pi_t &= (1-\tau)\left[(1-\alpha)\left(\frac{G_t}{N_t}\right)^{\theta} l_t^{\alpha} z_t^{1-\alpha} - \eta z_t - \xi\right] \\
&= (1-\tau)\left[(1-\alpha)\left(\frac{G_t}{N_t}\right)^{\theta} l_t^{\alpha}\Psi^{1-\alpha}\left(\frac{G_t}{N_t}\right)^{\frac{\theta(1-\alpha)}{\alpha}} l_t^{1-\alpha} - \eta\Psi\left(\frac{G_t}{N_t}\right)^{\frac{\theta}{\alpha}} l_t - \xi\right] \\
&= (1-\tau)\left[(1-\alpha)\Psi^{1-\alpha}\left(\frac{G_t}{N_t}\right)^{\frac{\theta}{\alpha}} l_t - \eta\Psi\left(\frac{G_t}{N_t}\right)^{\frac{\theta}{\alpha}} l_t - \xi\right] \\
&= (1-\tau)\left[\Psi^{1-\alpha}\left\{(1-\alpha) - \eta\Psi^{\alpha}\right\}\left(\frac{G_t}{N_t}\right)^{\frac{\theta}{\alpha}} LF(h_t^*) - \xi\right] \\
&= (1-\tau)\left[\Psi^{1-\alpha}\left\{(1-\alpha) - \eta\Psi^{\alpha}\right\}\bar{g}^{\frac{\theta}{\alpha-\theta}}\Psi^{\frac{\theta(1-\alpha)}{\alpha-\theta}} L^{\frac{\theta}{\alpha-\theta}} F(h_t^*)^{\frac{\theta}{\alpha-\theta}} LF(h_t^*) - \xi\right] \\
&= (1-\tau)\left[\Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}}\alpha(1-\alpha)\bar{g}^{\frac{\theta}{\alpha-\theta}} L^{\frac{\alpha}{\alpha-\theta}} F(h_t^*)^{\frac{\alpha}{\alpha-\theta}} - \xi\right] \\
&= (1-\tau_t)\left[\Pi(h_t^*;\bar{g}) - \xi\right]. \qquad \text{(B.14)}
\end{aligned}
$$

where

$$
\Pi(h_t^*;\bar{g}) \equiv \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}}\alpha(1-\alpha)\bar{g}^{\frac{\theta}{\alpha-\theta}} L^{\frac{\alpha}{\alpha-\theta}} F(h_t^*)^{\frac{\alpha}{\alpha-\theta}}. \qquad \text{(B.15)}
$$

From (B.13) and (B.14), we obtain

$$
\frac{\dot{\nu}_t}{\nu_t} = r_t - \frac{(1-\tau_t)\cdot\{\Pi(h_t^*;\bar{g}) - \xi\}}{\nu_t}. \qquad \text{(B.16)}
$$

Then, (B.11) implies

$$
\frac{1}{\nu_t} = \frac{\delta h_t^*}{\alpha\Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}}\bar{g}^{\frac{\theta}{\alpha-\theta}} L^{\frac{\theta}{\alpha-\theta}} F(h_t^*)^{\frac{\theta}{\alpha-\theta}}} = \frac{\delta h_t^*}{\dfrac{\Pi(h_t^*;\bar{g})}{(1-\alpha)LF(h_t^*)}} = \frac{(1-\alpha)LF(h_t^*)\delta h_t^*}{\Pi(h_t^*;\bar{g})}. \qquad \text{(B.17)}
$$

Inserting (B.17) into (B.16), we have

$$
\frac{\dot{\nu}_t}{\nu_t} = r_t - \frac{(1-\tau_t)(1-\alpha)LF(h_t^*)\delta h_t^*\left\{\Pi(h_t^*;\bar{g}) - \xi\right\}}{\Pi(h_t^*;\bar{g})}. \qquad \text{(B.18)}
$$

Inserting (B.18) into (B.12) yields

$$\left\{1 - \frac{\theta}{\alpha-\theta} \cdot \frac{F'(h_t^*)h_t^*}{F(h_t^*)}\right\} \cdot \frac{\dot{h}_t^*}{h_t^*} = -r_t + \frac{(1-\tau_t)(1-\alpha)LF(h_t^*)\delta h_t^* \left\{\Pi(h_t^*;\bar{g}) - \xi\right\}}{\Pi(h_t^*;\bar{g})}. \tag{B.19}$$

## B.3 Government budget constraint

Using (B.5) and (B.14), we can rewrite the government budget constraint in (15) as

$$\tau_t = \frac{\bar{g}^{\frac{\alpha}{\alpha-\theta}} \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} L^{\frac{\alpha}{\alpha-\theta}} F(h_t^*)^{\frac{\alpha}{\alpha-\theta}}}{\Pi(h_t^*;\bar{g}) - \xi} = \frac{\bar{g}}{\alpha(1-\alpha)\left(1 - \dfrac{\xi}{\Pi(h_t^*;\bar{g})}\right)}. \tag{B.20}$$

Note that Assumption 2 implies

$$\frac{\bar{g}}{\alpha(1-\alpha)\left(1 - \dfrac{\xi}{\Pi(h_{max};\bar{g})}\right)} < 1 \quad \text{and} \quad 1 - \frac{\xi}{\Pi(h_{max};\bar{g})} > 0, \tag{B.21}$$

Since $\Pi(h_t^*;\bar{g})$ is an increasing function of $h_t^*$, and $\Pi(h_{min};\bar{g}) = 0$, Assumption 2 ensures the existence of a $\underline{h}(\bar{g}) \in (h_{min}, h_{max})$, which satisfies the following condition:

$$\frac{\bar{g}}{\alpha(1-\alpha)\left(1 - \dfrac{\xi}{\Pi(\underline{h}(\bar{g});\bar{g})}\right)} = 1. \tag{B.22}$$

When $h_t^* \in [\underline{h}(\bar{g}), h_{max}]$, $\tau_t \in (0,1]$ is satisfied. In what follows, we concentrate on the case of $h_t^* \geq \underline{h}(\bar{g})$ in which the government budget is balanced.

## B.4 Equilibrium dynamics

Equilibrium dynamics is characterized by (B.9), (B.19), and (B.20). Eliminating $r_t$ from (B.9) and (B.19), we have

$$\begin{aligned}&\left[\frac{\sigma}{\widehat{c}_t} \cdot \frac{\Pi(h_t^*;\bar{g})}{(\alpha-\theta)(1-\alpha)L} \cdot \{\alpha(2-\alpha) - \bar{g}\} \cdot \frac{F'(h_t^*)h_t^*}{F(h_t^*)} + 1 - \frac{\theta}{\alpha-\theta} \cdot \frac{F'(h_t^*)h_t^*}{F(h_t^*)}\right] \cdot \frac{\dot{h}_t^*}{h_t^*} \\ &= \frac{(1-\tau_t)(1-\alpha)LF(h_t^*)\delta h_t^* \left\{\Pi(h_t^*;\bar{g}) - \xi\right\}}{\Pi(h_t^*;\bar{g})} - \{\rho + \sigma\phi(h_t^*)\}.\end{aligned} \tag{B.23}$$

Using the definition of $\Pi(h_t^*;\bar{g})$, we can rewrite equation (B.7) as

$$\widehat{c}_t = \frac{\Pi(h_t^*;\bar{g})\left\{\alpha(2-\alpha)-\bar{g}\right\}-\alpha(1-\alpha)\xi}{\alpha(1-\alpha)L}.$$

Inserting (B.24) into the left-hand side of (B.23), we have

$$\begin{aligned}
&\Bigg[\frac{\sigma\alpha(1-\alpha)L}{\Pi(h_t^*;\bar{g})\left\{\alpha(2-\alpha)-\bar{g}\right\}-\alpha(1-\alpha)\xi}\cdot\frac{\Pi(h_t^*;\bar{g})}{(\alpha-\theta)(1-\alpha)L}\cdot\left\{\alpha(2-\alpha)-\bar{g}\right\}\cdot\frac{F'(h_t^*)h_t^*}{F(h_t^*)}\\
&\qquad +1-\frac{\theta}{\alpha-\theta}\cdot\frac{F'(h_t^*)h_t^*}{F(h_t^*)}\Bigg]\cdot\frac{\dot{h}_t^*}{h_t^*}\\
&=\left\{1+\frac{1}{\alpha-\theta}\cdot\frac{F'(h_t^*)h_t^*}{F(h_t^*)}\cdot\left[\frac{\sigma\alpha\Pi(h_t^*;\bar{g})\cdot\left\{\alpha(2-\alpha)-\bar{g}\right\}}{\Pi(h_t^*;\bar{g})\left\{\alpha(2-\alpha)-\bar{g}\right\}-\alpha(1-\alpha)\xi}-\theta\right]\right\}\cdot\frac{\dot{h}_t^*}{h_t^*}\\
&=\left\{1+\frac{1}{\alpha-\theta}\cdot\frac{F'(h_t^*)h_t^*}{F(h_t^*)}\cdot\left[\frac{(\sigma\alpha-\theta)\Pi(h_t^*;\bar{g})\cdot\left\{\alpha(2-\alpha)-\bar{g}\right\}+\alpha\theta(1-\alpha)\xi}{\Pi(h_t^*;\bar{g})\left\{\alpha(2-\alpha)-\bar{g}\right\}-\alpha(1-\alpha)\xi}\right]\right\}\cdot\frac{\dot{h}_t^*}{h_t^*}\\
&=\Gamma(h_t^*;\bar{g})\cdot\frac{\dot{h}_t^*}{h_t^*}.
\end{aligned}\tag{B.24}$$

where

$$\Gamma(h_t^*;\bar{g})\equiv 1+\frac{1}{\alpha-\theta}\cdot\frac{F'(h_t^*)h_t^*}{F(h_t^*)}\cdot\left[\frac{(\sigma\alpha-\theta)\Pi(h_t^*;\bar{g})\cdot\left\{\alpha(2-\alpha)-\bar{g}\right\}+\alpha\theta(1-\alpha)\xi}{\Pi(h_t^*;\bar{g})\left\{\alpha(2-\alpha)-\bar{g}\right\}-\alpha(1-\alpha)\xi}\right].\tag{B.25}$$

Finally, inserting (B.24) into (B.23), we obtain

$$\begin{aligned}
&\Gamma(h_t^*;\bar{g})\cdot\frac{\dot{h}_t^*}{h_t^*}\\
&=\frac{(1-\tau_t)(1-\alpha)LF(h_t^*)\delta h_t^*\left\{\Pi(h_t^*;\bar{g})-\xi\right\}}{\Pi(h_t^*;\bar{g})}-\left\{\rho+\sigma\phi(h_t^*)\right\}\\
&=\frac{(1-\tau_t)(1-\alpha)LF(h_t^*)\delta\left\{\Pi(h_t^*;\bar{g})-\xi\right\}}{\Pi(h_t^*;\bar{g})}\cdot\left[h_t^*-\frac{\Pi(h_t^*;\bar{g})\{\rho+\sigma\phi(h_t^*)\}}{(1-\tau_t)(1-\alpha)LF(h_t^*)\delta\left\{\Pi(h_t^*;\bar{g})-\xi\right\}}\right]\\
&=\frac{(1-\tau_t)(1-\alpha)LF(h_t^*)\delta\left\{\Pi(h_t^*;\bar{g})-\xi\right\}}{\Pi(h_t^*;\bar{g})}\\
&\qquad\times\left[h_t^*-\frac{\rho+\sigma\phi(h_t^*)}{\frac{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h_t^*;\bar{g})}\right)-\bar{g}}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h_t^*;\bar{g})}\right)}\cdot(1-\alpha)LF(h_t^*)\delta\left(1-\frac{\xi}{\Pi(h_t^*;\bar{g})}\right)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(1-\tau_t)(1-\alpha)LF(h_t^*)\delta\left\{\Pi(h_t^*;\bar{g})-\xi\right\}}{\Pi(h_t^*;\bar{g})}\cdot\left[h_t^*-\frac{\rho+\sigma\phi(h_t^*)}{\left\{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h_t^*;\bar{g})}\right)-\bar{g}\right\}\cdot\frac{LF(h_t^*)\delta}{\alpha}}\right] \\
&= \frac{(1-\tau_t)(1-\alpha)LF(h_t^*)\delta\left\{\Pi(h_t^*;\bar{g})-\xi\right\}}{\Pi(h_t^*;\bar{g})}\cdot\left[h_t^*-RHS(h_t^*;\bar{g})\right], \qquad \text{(B.26)}
\end{aligned}
$$

where

$$
RHS(h_t^*;\bar{g}) \equiv \frac{\rho+\sigma\phi(h_t^*)}{\left\{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h_t^*;\bar{g})}\right)-\bar{g}\right\}\cdot\frac{LF(h_t^*)\delta}{\alpha}}. \qquad \text{(B.27)}
$$

Finally, we show that the steady-state threshold, $h^{*s}$, is characterized by $h^{*s} = RHS(h^{asts};\bar{g})$. Recall that we concentrate on the case in which the government budget is balanced, i.e., $h_t^* \geq \underline{h}(\bar{g})$ (see equation (B.22)). Then, the definition of $\underline{h}(\bar{g})$ implies

$$
\begin{aligned}
&\Pi(\underline{h}(\bar{g});\bar{g})\left\{\alpha(2-\alpha)-\bar{g}\right\}-\alpha(1-\alpha)\xi \\
&\quad = \alpha\Pi(\underline{h}(\bar{g});\bar{g})+\alpha(1-\alpha)\cdot\left\{\Pi(\underline{h}(\bar{g});\bar{g})-\xi\right\}-\Pi(\underline{h}(\bar{g});\bar{g})\bar{g} \\
&\quad = \alpha\Pi(\underline{h}(\bar{g});\bar{g}) \\
&\quad > 0.
\end{aligned}
$$

Therefore,

$$
\Pi(h_t^*;\bar{g})\left\{\alpha(2-\alpha)-\bar{g}\right\}-\alpha(1-\alpha)\xi > 0,
$$

holds for any $h_t^* \geq \underline{h}(\bar{g})$. Under Assumption 1, this implies $\Gamma(h_t^*;\bar{g}) > 0$. Moreover, (B.22) implies $\Pi(\underline{h}(\bar{g});\bar{g}) > \xi$. Since $\Pi(h_t^*;\bar{g})$ is an increasing function of $h_t^*$, $\Pi(h_t^*;\bar{g}) > \xi \ \forall h_t^* \in [\underline{h}(\bar{g}), h_{max}]$ holds. Therefore, under Assumptions 1 and 2, we have

$$
\operatorname{sign}\frac{\dot{h}_t^*}{h_t^*} = \operatorname{sign}\left\{h_t^*-RHS(h_t^*;\bar{g})\right\}.
$$

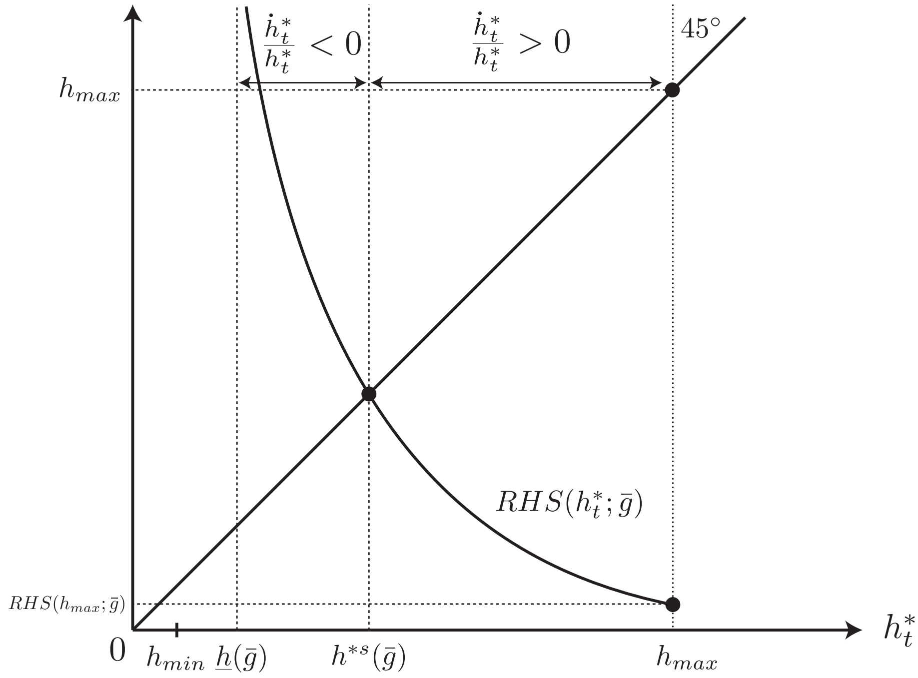


**Figure C.1:** Steady-state equilibrium.

# C Proof of Proposition 1

The steady-state threshold value of ability, $h^{*s}(\bar{g})$, satisfies (20). From (B.22) and (B.27), we obtain

$$\begin{aligned} \left.\frac{\partial RHS(h_t^*)}{\partial h_t^*}\right|_{h_t^* \in (\underline{h}(\bar{g}), h_{max})} &< 0, \\ \lim_{h_t^* \downarrow \underline{h}(\bar{g})} RHS(h_t^*;\bar{g}) &= +\infty. \end{aligned}$$

Therefore, if $h_{max} \geq RHS(h_{max};\bar{g})$, there exists a unique $h^{*s} \in [h_{min}, h_{max}]$ that satisfies $h_t^* = RHS(h^{*s}(\bar{g});\bar{g})$ (see Figure C.1).

Now, we derive the condition for $h_{max} \geq RHS(h_{max};\bar{g})$. Equation (B.27) implies that

$$h_{max} \geq RHS(h_{max};\bar{g}) \;\Leftrightarrow\; \Lambda(\bar{g}) \geq \frac{\alpha\rho}{L\delta h_{max}}, \tag{C.1}$$

where

$$\Lambda(\bar{g}) \equiv \alpha(1-\alpha) \cdot \left\{1 - \frac{\xi}{\Pi(h_{max};\bar{g})}\right\} - \bar{g}. \tag{C.2}$$

From (C.1), we have

$$\begin{aligned}
\Lambda'(\bar{g}) &= \alpha(1-\alpha)\cdot\frac{\xi}{\Pi(h_{max};\bar{g})^2}\cdot\frac{\theta}{\alpha-\theta}\cdot\frac{1}{\bar{g}}\cdot\Pi(h_{max};\bar{g})-1\\
&= \frac{\alpha(1-\alpha)\theta}{\alpha-\theta}\cdot\frac{\xi}{\Pi(h_{max};\bar{g})}\cdot\frac{1}{\bar{g}}-1,\\
\Lambda''(\bar{g}) &= -\frac{\alpha(1-\alpha)\theta}{\alpha-\theta}\cdot\frac{\xi}{\{\Pi(h_{max};\bar{g})\bar{g}\}^2}\cdot\left\{\Pi(h_{max};\bar{g})+\frac{\theta}{\alpha-\theta}\cdot\frac{1}{\bar{g}}\cdot\Pi(h_{max};\bar{g})\bar{g}\right\}\\
&= -\frac{\alpha(1-\alpha)\theta}{\alpha-\theta}\cdot\frac{\xi}{\Pi(h_{max};\bar{g})\bar{g}^2}\cdot\left(1+\frac{\theta}{\alpha-\theta}\right)\\
&= -\frac{\alpha^2(1-\alpha)\theta}{(\alpha-\theta)^2}\cdot\frac{\xi}{\Pi(h_{max};\bar{g})\bar{g}^2}\\
&< 0.
\end{aligned}$$

Let us denote the maximizer of $\Lambda(\bar{g})$ by $\bar{g}^*$, which is given by

$$\begin{aligned}
\Lambda'(\bar{g}^*)=0 \quad &\Leftrightarrow \quad \frac{\alpha(1-\alpha)\theta}{\alpha-\theta}\cdot\frac{\xi}{\Pi(h_{max};\bar{g}^*)}\cdot\frac{1}{\bar{g}^*}-1=0\\
&\Leftrightarrow \quad \Pi(h_{max};\bar{g}^*)\bar{g}^* = \frac{\alpha(1-\alpha)\theta\xi}{\alpha-\theta}\\
&\Leftrightarrow \quad \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}}\alpha(1-\alpha)(\bar{g}^*)^{\frac{\alpha}{\alpha-\theta}}L^{\frac{\alpha}{\alpha-\theta}} = \frac{\alpha(1-\alpha)\theta\xi}{\alpha-\theta}\\
&\Leftrightarrow \quad \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}}(\bar{g}^*)^{\frac{\alpha}{\alpha-\theta}}L^{\frac{\alpha}{\alpha-\theta}} = \frac{\theta\xi}{\alpha-\theta}\\
&\Leftrightarrow \quad \Psi^{1-\alpha}\bar{g}^*L = \left(\frac{\theta\xi}{\alpha-\theta}\right)^{\frac{\alpha-\theta}{\alpha}}\\
&\Leftrightarrow \quad \bar{g}^* = \frac{1}{\Psi^{1-\alpha}L}\cdot\left(\frac{\theta\xi}{\alpha-\theta}\right)^{\frac{\alpha-\theta}{\alpha}}.
\end{aligned}$$

Therefore, $\Lambda(\bar{g})$ has a unique maximum point.

Using (C.2), we obtain

$$\begin{aligned}
\lim_{\bar{g}\to 0}\Lambda(\bar{g}) &= \alpha(1-\alpha)\left\{1-\frac{\xi}{\lim_{\bar{g}\to 0}\Pi(h_{max};\bar{g})}\right\} = -\infty,\\
\Lambda(1) &= \alpha(1-\alpha)\left\{1-\frac{\xi}{\Pi(h_{max};1)}\right\}-1<0,
\end{aligned}$$

Assumption 3 ensures $\Lambda(\bar{g}^*) > \alpha\rho/L\delta h_{max}$. Therefore, there exists a unique $0 < \bar{g}_{min} <$

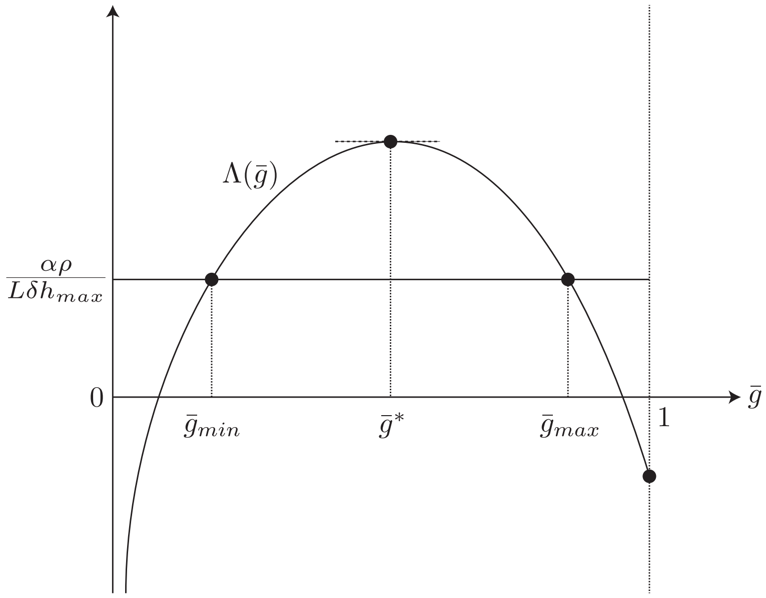


**Figure C.2:** The range of $\bar{g}$ in which the growth rate is positive.

$\bar{g}_{max} < 1$ which satisfies $RHS(h_{max}; \bar{g}_{min}) = RHS(h_{max}; \bar{g}_{max}) = h_{max}$ or $h^{*s}(\bar{g}_{min}) = h^{*s}(\bar{g}_{max}) = h_{max}$ (see Figure C.2). Finally, equations (13) and (C.1) imply that $h^{*s}(\bar{g}) < h_{max}$ and $\phi(h^{*s}) > 0$ for any $\bar{g} \in (\bar{g}_{min}, \bar{g}_{max})$.

The above shows that there exists a unique steady-state threshold value of ability, $h^{*s}(\bar{g})$, for any $\bar{g} \in (\bar{g}_{min}, \bar{g}_{max})$. Since equation (20) implies that the steady state is unstable, the economy is always in the steady state equilibrium. Since $h^{*s}(\bar{g}) > \underline{h}(\bar{g})$, $\tau \in [0, 1)$ is satisfied in the steady-state.

**Q.E.D.**

# D Proof of Lemma 1

## D.1 The derivation of equation (23)

Differentiating (13) with respect to $\bar{g}$ yields

$$\frac{\partial \phi(h^{*s})}{\partial \bar{g}} = -\delta L h^{*s} F'(h^{*s}) \cdot \frac{dh^{*s}}{d\bar{g}}. \tag{D.1}$$

From (22), we have

$$\frac{\mathrm{d}h^{*s}}{\mathrm{d}\bar{g}} = -\frac{-\dfrac{\partial RHS(h^{*s};\bar{g})}{\partial \bar{g}}}{1-\dfrac{\partial RHS(h^{*s};\bar{g})}{\partial h^{*s}}}. \tag{D.2}$$

Using (B.27), we obtain

$$\begin{aligned}
&\frac{\partial RHS(h^{*s};\bar{g})}{\partial \bar{g}} \\
&\quad = -\frac{\rho+\sigma\phi(h^{*s})}{\left\{\alpha(1-\alpha)\left(1-\dfrac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}\right\}^2\cdot\dfrac{LF(h^{*s})\delta}{\alpha}} \\
&\qquad\qquad \times\left\{\alpha(1-\alpha)\cdot\frac{\xi}{\Pi(h^{*s};\bar{g})^2}\cdot\frac{\theta}{\alpha-\theta}\cdot\frac{1}{\bar{g}}\cdot\Pi(h^{*s};\bar{g})-1\right\} \\
&\quad = -\frac{RHS(h^{*s};\bar{g})}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}}\cdot\left\{\frac{\alpha(1-\alpha)\xi\theta}{\alpha-\theta}\cdot\frac{1}{\Pi(h^{*s};\bar{g})\bar{g}}-1\right\}.
\end{aligned} \tag{D.3}$$

Using the definition of $\bar{g}^*$ in Assumption 3, we obtain

$$\begin{aligned}
\frac{\alpha(1-\alpha)\xi\theta}{\alpha-\theta}\cdot\frac{1}{\Pi(h^{*s};\bar{g})\bar{g}} &= \frac{\alpha(1-\alpha)\xi\theta}{\alpha-\theta}\cdot\frac{1}{\alpha(1-\alpha)\left\{\Psi^{1-\alpha}L\bar{g}F(h^{*s})\right\}^{\frac{\alpha}{\alpha-\theta}}} \\
&= \frac{1}{\left\{\bar{g}F(h^{*s})\right\}^{\frac{\alpha}{\alpha-\theta}}}\cdot\frac{1}{\left\{\Psi^{1-\alpha}L\right\}^{\frac{\alpha}{\alpha-\theta}}}\cdot\frac{\xi\theta}{\alpha-\theta} \\
&= \left(\frac{\bar{g}^*}{\bar{g}F(h^{*s})}\right)^{\frac{\alpha}{\alpha-\theta}}.
\end{aligned} \tag{D.4}$$

Inserting (22) and (D.4) yields

$$\frac{\partial RHS(h^{*s};\bar{g})}{\partial \bar{g}} = -h^{*s}\cdot\frac{\left(\frac{\bar{g}^*}{\bar{g}F(h^{*s})}\right)^{\frac{\alpha}{\alpha-\theta}}-1}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}}. \tag{D.5}$$

Next, differentiating (B.27) with respect to $\bar{g}$ yields

$$\frac{\partial RHS(h^{*s};\bar{g})}{\partial h^{*s}}$$

$$
\begin{aligned}
&= \frac{1}{\left\{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}\right\}^2\cdot\left(\frac{LF(h^{*s})\delta}{\alpha}\right)^2} \\
&\quad\times\left[\sigma(-1)\delta Lh^{*s}F'(h^{*s})\cdot\left\{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}\right\}\cdot\frac{LF(h^{*s})\delta}{\alpha}\right. \\
&\qquad-(\rho+\sigma\phi(h^{*s}))\cdot\left\{\alpha(1-\alpha)\cdot\frac{\xi}{\Pi(h^{*s};\bar{g})^2}\cdot\frac{\alpha}{\alpha-\theta}\cdot\frac{F'(h^{*s})}{F(h^{*s})}\cdot\Pi(h^{*s};\bar{g})\cdot\frac{LF(h^{*s})\delta}{\alpha}\right. \\
&\qquad\left.\left.+\left\{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}\right\}\cdot\frac{LF'(h^{*s})\delta}{\alpha}\right\}\right] \\
&= -\frac{1}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}} \\
&\times\left[\sigma\delta h^{*s}LF'(h^{*s})\cdot\frac{\alpha}{LF(h^{*s})\delta}\right. \\
&\qquad+RHS(h^{*s};\bar{g})\frac{\alpha}{LF(h^{*s})\delta}\cdot\left\{\frac{\alpha(1-\alpha)\xi}{\alpha-\theta}\cdot\frac{L\delta}{\Pi(h^{*s};\bar{g})}\cdot F'(h^{*s})\right. \\
&\qquad\qquad\left.\left.+\left\{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}\right\}\cdot\frac{LF'(h^{*s})\delta}{\alpha}\right\}\right] \\
&= -\frac{\frac{h^{*s}F'(h^{*s})}{F(h^{*s})}}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}} \\
&\quad\times\left[\alpha\sigma+\alpha\cdot\frac{\alpha(1-\alpha)\xi}{\alpha-\theta}\cdot\frac{1}{\Pi(h^{*s};\bar{g})}+\left\{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}\right\}\right] \\
&= -\frac{\frac{h^{*s}F'(h^{*s})}{F(h^{*s})}}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}}\cdot\left[\alpha\sigma+\alpha(1-\alpha)+\frac{\alpha(1-\alpha)\xi\theta}{\alpha-\theta}\cdot\frac{1}{\Pi(h^{*s};\bar{g})}-\bar{g}\right].
\end{aligned}
\tag{D.6}
$$

Inserting (D.4) into (D.6), we obtain

$$
\frac{\partial RHS(h^{*s};\bar{g})}{\partial h^{*s}} = -\frac{\frac{h^{*s}F'(h^{*s})}{F(h^{*s})}\cdot\left[\alpha\sigma+\alpha(1-\alpha)+\bar{g}\cdot\left\{\left(\frac{\bar{g}^*}{\bar{g}F(h^{*s})}\right)^{\frac{\alpha}{\alpha-\theta}}-1\right\}\right]}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}}. \tag{D.7}
$$

Finally, inserting (D.2), (D.5), and (D.7) into (D.1), we obtain

$$
\frac{\mathrm{d}\phi(h^{*s})}{\mathrm{d}\bar{g}} \quad = \quad \delta Lh^{*s}F'(h^{*s})\cdot\frac{h^{*s}\cdot\frac{\left(\frac{\bar{g}^*}{\bar{g}F(h^{*s})}\right)^{\frac{\alpha}{\alpha-\theta}}-1}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}}}{1+\frac{\frac{h^{*s}F'(h^{*s})}{F(h^{*s})}\cdot\left[\alpha\sigma+\alpha(1-\alpha)+\bar{g}\cdot\left\{\left(\frac{\bar{g}^*}{\bar{g}F(h^{*s})}\right)^{\frac{\alpha}{\alpha-\theta}}-1\right\}\right]}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi(h^{*s};\bar{g})}\right)-\bar{g}}}
$$

$$= \frac{\delta L h^{*s} F(h^{*s}) \cdot \left\{ \left( \frac{\bar{g}^*}{\bar{g} F(h^{*s})} \right)^{\frac{\alpha}{\alpha-\theta}} - 1 \right\}}{\frac{F(h^{*s})}{h^{*s} F'(h^{*s})} \cdot \left[ \alpha(1-\alpha) \left( 1 - \frac{\xi}{\Pi(h^{*s};\bar{g})} \right) - \bar{g} \right] + \left[ \alpha\sigma + \alpha(1-\alpha) + \bar{g} \cdot \left\{ \left( \frac{\bar{g}^*}{\bar{g} F(h^{*s})} \right)^{\frac{\alpha}{\alpha-\theta}} - 1 \right\} \right]}.$$

## D.2 An inverted U-shaped relationship between $\bar{g}$ and $\phi(h^{*s})$

Next, we show that there is an inverted U-shaped relationship between $\bar{g}$ and $\phi(h^{*s})$. Since Assumption 2 ensures $\alpha(1-\alpha) - \bar{g} > 0$, the denominator of the right-hand side of (23) is positive. Therefore, we have

$$\frac{\mathrm{d}\phi(h^{*s}(\bar{g}))}{\mathrm{d}\bar{g}} \gtreqless 0 \quad \Leftrightarrow \quad \bar{g}^* \gtreqless \bar{g} F(h^{*s}). \tag{D.8}$$

Since $h^{*s}(\bar{g}_{min}) = h^{*s}(\bar{g}_{max}) = h_{max}$ (see Lemma 1), we have

$$\begin{aligned} \bar{g}_{min} F(h^{*s}(\bar{g}_{min})) &= \bar{g}_{min} F(h_{max}) = \bar{g}_{min} < \bar{g}^*, \\ \bar{g}_{max} F(h^{*s}(\bar{g}_{max})) &= \bar{g}_{max} F(h_{max}) = \bar{g}_{max} > \bar{g}^*. \end{aligned}$$

Therefore, there exists at least one $\bar{g}^{thres} \in (\bar{g}_{min}, \bar{g}_{max})$ such that $\bar{g}^* = \bar{g}^{thres} F(h^{*s}(\bar{g}^{thres}))$ (see Figure D.1).

Next, we prove the uniqueness of $\bar{g}^{thres}$. Note that (13) and the definition of $\bar{g}^{thres}$ implies $dh^{*s}(\bar{g})/d\bar{g}|_{\bar{g}=\bar{g}^{thres}} = 0$. Therefore, we obtain

$$\begin{aligned} \frac{\mathrm{d}\left\{\bar{g} F(h^{*s}(\bar{g}))\right\}}{\mathrm{d}\bar{g}}\Big|_{\bar{g}=\bar{g}^{thres}} &= F(h^{*s}(\bar{g}^{thres})) + \bar{g}^{thres} F'(h^{*s}(\bar{g}^{thres})) \cdot \frac{\mathrm{d}h^{*s}(\bar{g})}{\mathrm{d}\bar{g}}\Big|_{\bar{g}=\bar{g}^{thres}} \\ &= F(h^{*s}(\bar{g}^{thres})) \\ &> 0. \end{aligned}$$

Therefore, $\bar{g}^{thres}$, which satisfies $\bar{g}^* = \bar{g}^{thres} F(h^{*s}(\bar{g}^{thres}))$, is unique. Moreover, $h^{*s}(\bar{g})$ is a U-shaped function of $\bar{g}$ (see Figure D.2). Finally, (13) implies that the growth rate of $N_t$, $\phi(h^{*s}(\bar{g}))$, is a monotonically decreasing function of $h^{*s}(\bar{g})$. Therefore, we can conclude that an

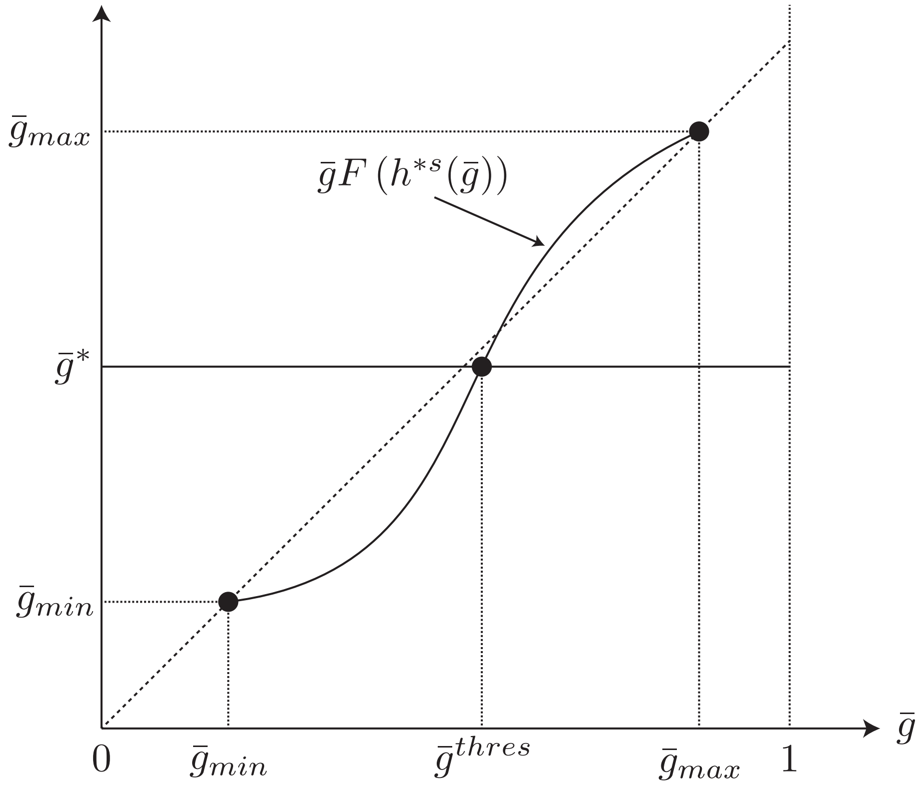


**Figure D.1:** Existence and uniqueness of $\bar{g}^{thresh}$.

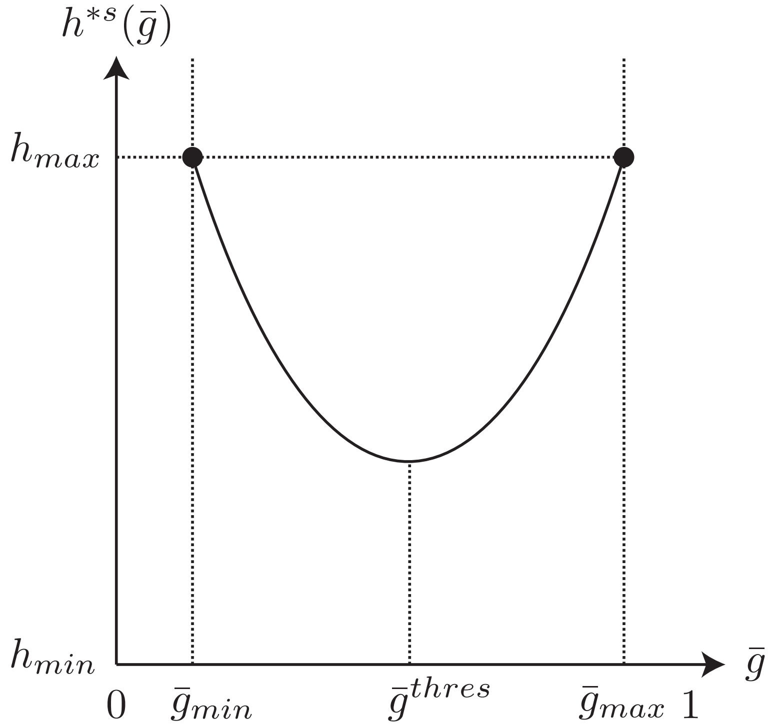


**Figure D.2:** U-shaped relationship between $\bar{g}$ and $h^{*s}(\bar{g})$.

inverted U-shaped relationship exists between $\bar{g}$ and $\phi(h^{*s}(\bar{g}))$.

**Q.E.D.**

# E Equilibrium dynamics in the homogeneous-ability economy

## E.1 The Euler equation

We retain the notation used in the heterogeneous-ability model as far as possible. Most of the first-order conditions in a heterogeneous-ability model can be applied to the homogeneous-ability model. Note that $F(h_t^*)$ in the heterogeneous-ability economy is replaced by $q_t$. When $F(h_t^*) = q_t$ holds, we obtain

$$\frac{\dot{q}_t}{q_t} = \frac{F'(h_t^*)\dot{h}_t^*}{F(h_t^*)}.$$

Then, the Euler equation in (B.9) can be rewritten as

$$\frac{1}{\sigma} \cdot (r_t^H - \rho) - \phi^H(q_t) = \frac{1}{\widehat{c}_t^H} \cdot \frac{\alpha \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} L^{\frac{\theta}{\alpha-\theta}} q_t^{\frac{\alpha}{\alpha-\theta}} \bar{g}^{\frac{\theta}{\alpha-\theta}}}{\alpha - \theta} \cdot \{\alpha(2-\alpha) - \bar{g}\} \cdot \frac{\dot{q}_t}{q_t}. \tag{E.1}$$

where $\phi^H(q^*) = \delta L \widehat{h}(1 - q^*)$.

## E.2 Occupational choice

In an equilibrium with both workers and entrepreneurs (i.e., $q_t \in (0, 1)$), the threshold condition in (B.11) is replaced by

$$\alpha \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} \bar{g}^{\frac{\theta}{\alpha-\theta}} L^{\frac{\theta}{\alpha-\theta}} q_t^{\frac{\theta}{\alpha-\theta}} = \nu_t^H \delta \widehat{h}, \tag{E.2}$$

where $\nu_t^H$ is a ratio of the co-state variable associated with the budget constraint and the co-state variable associated with law of motion, i.e., $\nu_t^H \equiv \zeta_t^H / \lambda_t^H$.

Taking logarithms and differentiating both sides of (E.2) yields

$$\frac{\theta}{\alpha-\theta}\cdot\frac{\dot{q}_t}{q_t}=\frac{\dot{\nu}_t}{\nu_t}=\frac{\dot{\zeta}_t^H}{\zeta_t^H}-\frac{\dot{\lambda}_t^H}{\lambda_t^H}, \tag{E.3}$$

where the second equality comes from the definition of $\nu_t^H$.

When $F(h_t^*)=q_t$ holds, (B.18) can be rewritten as

$$\frac{\dot{\nu}_t}{\nu_t}=r_t^H-\frac{(1-\tau_t^H)(1-\alpha)Lq_t\delta\widehat{h}\left\{\Pi^H(q_t;\bar{g})-\xi\right\}}{\Pi^H(q_t;\bar{g})}. \tag{E.4}$$

where

$$\Pi^H(q_t;\bar{g})=\Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}}\alpha(1-\alpha)\bar{g}^{\frac{\theta}{\alpha-\theta}}L^{\frac{\alpha}{\alpha-\theta}}q_t^{\frac{\alpha}{\alpha-\theta}}.$$

Inserting (E.4) into (E.3), we obtain

$$\frac{\theta}{\alpha-\theta}\cdot\frac{\dot{q}_t}{q_t}=r_t^H-\frac{(1-\tau_t^H)(1-\alpha)Lq_t\delta\widehat{h}\left\{\Pi^H(q_t;\bar{g})-\xi\right\}}{\Pi^H(q_t;\bar{g})}. \tag{E.5}$$

Then, (B.20) is replaced by

$$\tau_t^H=\frac{\bar{g}}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q_t;\bar{g})}\right)}. \tag{E.6}$$

Since $\Pi(h_{max};\bar{g})=\Pi^H(1;\bar{g})$, Assumption 2 implies $\Pi^H(1;\bar{g})>\xi$ and

$$\frac{\bar{g}}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(1;\bar{g})}\right)}<1.$$

Recall that $\Pi^H(q_t;\bar{g})$ is an increasing function and $\Pi^H(0;\bar{g})=0$. Therefore, Assumption 2 ensures the existence of $\underline{q}(\bar{g}) \in (0,1)$, which satisfies the following condition:

$$\frac{\bar{g}}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(\underline{q}(\bar{g});\bar{g})}\right)} = 1, \tag{E.7}$$

when $q_t \in [\underline{q}(\bar{g}), 1]$ and $\tau_t^H \in [0,1)$ hold. In what follows, we concentrate on the case of $q_t \geq \underline{q}(\bar{g})$ in which the government budget is balanced.

## E.3 Equilibrium dynamics

From (E.1), (E.5), and (E.6), we obtain

$$\Gamma^H(q_t;\bar{g}) \cdot \frac{\dot{q}_t}{q_t} = \frac{(1-\tau_t^H)(1-\alpha)Lq_t\delta\left\{\Pi^H(q_t;\bar{g})-\xi\right\}}{\Pi^H(q_t;\bar{g})} \cdot \left[\widehat{h} - RHS^H(q_t;\bar{g})\right],$$

where

$$\begin{aligned}
RHS^H(q_t;\bar{g}) &\equiv \frac{\rho + \sigma\phi^H(q_t)}{\left\{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q_t;\bar{g})}\right)-\bar{g}\right\}\cdot\frac{Lq_t\delta}{\alpha}},\\
\Pi^H(q_t;\bar{g}) &\equiv \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}}\alpha(1-\alpha)\bar{g}^{\frac{\theta}{\alpha-\theta}}L^{\frac{\alpha}{\alpha-\theta}}q_t^{\frac{\alpha}{\alpha-\theta}},\\
\Gamma^H(q_t;\bar{g}) &\equiv \frac{1}{\alpha-\theta}\cdot\left[\frac{(\sigma\alpha-\theta)\Pi^H(q_t;\bar{g})\cdot\{\alpha(2-\alpha)-\bar{g}\}+\alpha\theta(1-\alpha)\xi}{\Pi^H(q_t;\bar{g})\{\alpha(2-\alpha)-\bar{g}\}-\alpha(1-\alpha)\xi}\right] > 0.
\end{aligned}$$

Similar to our approach in Appendix B, we can obtain $\Gamma^H(q_t;\bar{g}) > 0$ and $\Pi^H(q_t;\bar{g}) > \xi \ \forall q_t \in [\underline{q}(\bar{g}), 1]$. Therefore, under Assumptions 1 and 2, we have

$$\text{sign}\frac{\dot{q}_t}{q_t} = \text{sign}\left\{\widehat{h} - RHS^H(q_t;\bar{g})\right\}. \tag{E.8}$$

The equilibrium fraction of workers in the homogeneous-ability economy is characterized by

$$\widehat{h} = RHS^H(q^*;\bar{g}). \tag{E.9}$$

**Q.E.D.**

# F Proof of Proposition 2

Here we similarly follow the proof of Proposition 1. The steady-state fraction of workers, $q^*$, satisfies (24). From (25) and (E.7), we obtain

$$\left.\frac{\partial RHS^H(q_t)}{\partial q_t}\right|_{q_t \in (\underline{q}(\bar{g}),1)} < 0,$$
$$\lim_{q_t \downarrow \underline{q}(\bar{g})} RHS^H(q_t;\bar{g}) = +\infty.$$

Therefore, if $\widehat{h} \geq RHS^H(1;\bar{g})$, there exists a unique $q^* \in [0,1]$ that satisfies $\widehat{h} = RHS^H(q^*;\bar{g})$.

Next, we derive the condition for $\widehat{h} \geq RHS^H(1;\bar{g})$. Note that $\Pi(h_{max};\bar{g}) = \Pi^H(1;\bar{g})$ holds. Then, (25) implies

$$\widehat{h} \geq RHS^H(1;\bar{g}) \;\Leftrightarrow\; \Lambda(\bar{g}) \geq \frac{\alpha\rho}{L\delta\widehat{h}}, \tag{F.1}$$

where the definition of $\Lambda(\bar{g})$ is given in (C.2). Then, Assumption 4 ensures $\Lambda(\bar{g}^*) > \alpha\rho/L\delta\widehat{h}$. Therefore, there exists a unique $0 < \bar{g}^H_{min} < \bar{g}^H_{max} < 1$ that satisfies $RHS^H(1;\bar{g}^H_{min}) = RHS^H(1;\bar{g}^H_{max}) = \widehat{h}$ or $q^*(\bar{g}^H_{min}) = q^*(\bar{g}^H_{max}) = 1$. Finally, (25) and (E.9) imply that $q^*(\bar{g}) < 1$ for any $\bar{g} \in (\bar{g}^H_{min}, \bar{g}^H_{max})$, and it implies that the growth rate of $N_t$, $\phi(h^{*s})$, is strictly positive for any $\bar{g} \in (\bar{g}^H_{min}, \bar{g}^H_{max})$.

The above shows that there exists a steady-state fraction of workers, $q^*(\bar{g})$, for any $\bar{g} \in (\bar{g}^H_{min}, \bar{g}^H_{max})$. Since equation (E.8) implies that the steady state is unstable, the economy is always in the steady-state equilibrium. Since $q^*(\bar{g}) > \underline{q}(\bar{g})$, $\tau^H \in [0,1)$ is satisfied in the steady state.

**Q.E.D.**

# G Proof of Lemma 2

Since the steady-state growth rate is given by $\phi^H(q^*) = \delta L\widehat{h}(1-q^*)$, we obtain

$$\frac{\partial \phi^H(q^*)}{\partial \bar{g}} = -\delta L\widehat{h} \cdot \frac{dq^*}{d\bar{g}}. \tag{G.1}$$

From (E.9), we have

$$\frac{dq^*}{d\bar{g}} = -\frac{\dfrac{-\partial RHS^H(q^*;\bar{g})}{\partial \bar{g}}}{-\dfrac{\partial RHS^H(q^*;\bar{g})}{\partial q^*}}, \tag{G.2}$$

Using (25), we obtain

$$\begin{aligned}
\frac{\partial RHS^H(q^*;\bar{g})}{\partial \bar{g}} &= -\frac{\rho + \sigma\phi^H(q^*)}{\left\{\alpha(1-\alpha)\left(1-\dfrac{\xi}{\Pi^H(q^*;\bar{g})}\right)-\bar{g}\right\}^2 \cdot \dfrac{Lq^*\delta}{\alpha}} \\
&\qquad \times \left\{\alpha(1-\alpha)\cdot\frac{\xi}{\Pi^H(q^*;\bar{g})^2}\cdot\frac{\theta}{\alpha-\theta}\cdot\frac{1}{\bar{g}}\cdot\Pi^H(q^*;\bar{g})-1\right\} \\
&= -\frac{RHS^H(q^*;\bar{g})}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q^*;\bar{g})}\right)-\bar{g}} \cdot \left\{\frac{\alpha(1-\alpha)\xi\theta}{\alpha-\theta}\cdot\frac{1}{\Pi^H(q^*;\bar{g})\bar{g}}-1\right\}.
\end{aligned} \tag{G.3}$$

Using the definition of $\bar{g}^*$ in Assumption 3, we have

$$\begin{aligned}
\frac{\alpha(1-\alpha)\xi\theta}{\alpha-\theta}\cdot\frac{1}{\Pi^H(q^*;\bar{g})\bar{g}} &= \frac{\alpha(1-\alpha)\xi\theta}{\alpha-\theta}\cdot\frac{1}{\alpha(1-\alpha)\left\{\Psi^{1-\alpha}L\bar{g}q^*\right\}^{\frac{\alpha}{\alpha-\theta}}} \\
&= \frac{1}{\{\bar{g}q^*\}^{\frac{\alpha}{\alpha-\theta}}}\cdot\frac{1}{\{\Psi^{1-\alpha}L\}^{\frac{\alpha}{\alpha-\theta}}}\cdot\frac{\xi\theta}{\alpha-\theta} \\
&= \left(\frac{\bar{g}^*}{\bar{g}q^*}\right)^{\frac{\alpha}{\alpha-\theta}},
\end{aligned} \tag{G.4}$$

Inserting (G.4) into (G.3) yields

$$\frac{\partial RHS^H(q^*;\bar{g})}{\partial \bar{g}} = -\widehat{h}\cdot \frac{\left(\frac{\bar{g}^*}{\bar{g}q^*}\right)^{\frac{\alpha}{\alpha-\theta}} - 1}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q^*;\bar{g})}\right)-\bar{g}}. \tag{G.5}$$

Next, differentiating (25) with respect to $\bar{g}$ yields

$$\begin{aligned}
&\frac{\partial RHS^H(q^*;\bar{g})}{\partial q^*}\\
&= \frac{1}{\left\{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q^*;\bar{g})}\right)-\bar{g}\right\}^2\cdot\left(\frac{Lq^*\delta}{\alpha}\right)^2}\\
&\quad\times\Bigg[\sigma(-1)\delta L\widehat{h}\cdot\left\{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q^*;\bar{g})}\right)-\bar{g}\right\}\cdot\frac{Lq^*\delta}{\alpha}\\
&\qquad -(\rho+\sigma\phi^H(q^*))\cdot\Bigg\{\alpha(1-\alpha)\cdot\frac{\xi}{\Pi^H(q^*;\bar{g})^2}\cdot\frac{\alpha}{\alpha-\theta}\cdot\frac{1}{q^*}\cdot\Pi^H(q^*;\bar{g})\cdot\frac{Lq^*\delta}{\alpha}\\
&\qquad +\left\{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q^*;\bar{g})}\right)-\bar{g}\right\}\cdot\frac{L\delta}{\alpha}\Bigg\}\Bigg]\\
&= -\frac{1}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q^*;\bar{g})}\right)-\bar{g}}\\
&\quad\times\Bigg[\sigma\delta\widehat{h}L\cdot\frac{\alpha}{Lq^*\delta}\\
&\qquad +RHS^H(q^*;\bar{g})\frac{\alpha}{Lq^*\delta}\cdot\Bigg\{\frac{\alpha(1-\alpha)\xi}{\alpha-\theta}\cdot\frac{L\delta}{\Pi^H(q^*;\bar{g})}\\
&\qquad\qquad +\left\{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q^*;\bar{g})}\right)-\bar{g}\right\}\cdot\frac{L\delta}{\alpha}\Bigg\}\Bigg]\\
&= -\frac{\frac{\widehat{h}}{q^*}}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q^*;\bar{g})}\right)-\bar{g}}\\
&\qquad\times\left[\alpha\sigma+\alpha\cdot\frac{\alpha(1-\alpha)\xi}{\alpha-\theta}\cdot\frac{1}{\Pi^H(q^*;\bar{g})}+\left\{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q^*;\bar{g})}\right)-\bar{g}\right\}\right]\\
&= -\frac{\frac{\widehat{h}}{q^*}}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q^*;\bar{g})}\right)-\bar{g}}\cdot\left[\alpha\sigma+\alpha(1-\alpha)+\frac{\alpha(1-\alpha)\xi\theta}{\alpha-\theta}\cdot\frac{1}{\Pi^H(q^*;\bar{g})}-\bar{g}\right].
\end{aligned}$$

Inserting (G.4) into (G.6), we obtain

$$\frac{\partial RHS^H(q^*;\bar{g})}{\partial q^*} = -\frac{\frac{\widehat{h}}{q^*} \cdot \left[\alpha\sigma + \alpha(1-\alpha) + \bar{g} \cdot \left\{ \left(\frac{\bar{g}^*}{\bar{g}q^*}\right)^{\frac{\alpha}{\alpha-\theta}} - 1 \right\}\right]}{\alpha(1-\alpha)\left(1 - \frac{\xi}{\Pi^H(q^*;\bar{g})}\right) - \bar{g}}. \tag{G.6}$$

Finally, inserting (G.2 ), (G.5), and (G.6) into (G.1), we obtain

$$\begin{aligned}
\frac{d\phi^H(q^*)}{d\bar{g}} &= \delta L\widehat{h} \cdot \frac{\widehat{h} \cdot \frac{\left(\frac{\bar{g}^*}{\bar{g}q^*}\right)^{\frac{\alpha}{\alpha-\theta}} - 1}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q^*;\bar{g})}\right)-\bar{g}}}{\frac{\frac{\widehat{h}}{q^*}\cdot\left[\alpha\sigma+\alpha(1-\alpha)+\bar{g}\cdot\left\{\left(\frac{\bar{g}^*}{\bar{g}q^*}\right)^{\frac{\alpha}{\alpha-\theta}}-1\right\}\right]}{\alpha(1-\alpha)\left(1-\frac{\xi}{\Pi^H(q^*;\bar{g})}\right)-\bar{g}}} \\
&= \frac{\delta L\widehat{h}q^* \cdot \left\{ \left(\frac{\bar{g}^*}{\bar{g}q^*}\right)^{\frac{\alpha}{\alpha-\theta}} - 1 \right\}}{\left[\alpha\sigma + \alpha(1-\alpha) + \bar{g} \cdot \left\{ \left(\frac{\bar{g}^*}{\bar{g}q^*}\right)^{\frac{\alpha}{\alpha-\theta}} - 1 \right\}\right]}.
\end{aligned} \tag{G.7}$$

Next, we show that there is an inverted U-shaped relationship between $\bar{g}$ and $\phi^H(q^*(\bar{g}))$. Since Assumption 2 ensures $\alpha(1-\alpha) > \bar{g}$, the denominator of the right-hand side of (G.7) is positive. Therefore, we have

$$\frac{d\phi^H(q^*)}{d\bar{g}} \gtreqless 0 \Leftrightarrow \bar{g}^* \gtreqless \bar{g}q^*. \tag{G.8}$$

Then, we show the existence and uniqueness of $\bar{g}^{thres,H} \in (\bar{g}^H_{min}, \bar{g}^H_{max})$ such that $\bar{g}^* = \bar{g}^{thres,H}q^*(\bar{g}^{thres,H})$. Since $q^*(\bar{g}^H_{min}) = q^*(\bar{g}^H_{max}) = 1$, we obtain

$$\begin{aligned}
\bar{g}^H_{min}q^*(\bar{g}^H_{min}) &= \bar{g}^H_{min} < \bar{g}^*, \\
\bar{g}^H_{max}q^*(\bar{g}^H_{max}) &= \bar{g}^H_{max} > \bar{g}^*,
\end{aligned}$$

Therefore, the intermediate value theorem implies that there exists at least one $\bar{g}^{thres,H} \in (\bar{g}^H_{min}, \bar{g}^H_{max})$ such that $\bar{g}^* = \bar{g}^{thres,H}q^*(\bar{g}^{thres,H})$.

From the definition of $\bar{g}^{thres,H}$, we have

$$\begin{aligned}\left.\frac{d\{\bar{g}q^*(\bar{g})\}}{d\bar{g}}\right|_{\bar{g}=\bar{g}^{thres,H}} &= q^*(\bar{g}^{thres,H}) + \bar{g}^{thres,H}\cdot\left.\frac{dq^*(\bar{g})}{d\bar{g}}\right|_{\bar{g}=\bar{g}^{thres,H}} \\ &= q^*(\bar{g}^{thres,H}) \\ &> 0.\end{aligned}$$

Therefore, $\bar{g}^{thres,H}$ is unique and there exists an inverted U-shaped relationship between $\bar{g}$ and $\phi^H(q^*(\bar{g}))$.

**Q.E.D.**

# H Proof of Proposition 3

Note that $h_{max} > \widehat{h}$ implies $\alpha\rho/(L\delta\widehat{h}) > \alpha\rho/(L\delta h_{max})$. Therefore, from (C.1), (F.1), and Figure 3, we obtain $\bar{g}_{min} < \bar{g}^H_{min}$ and $\bar{g}^H_{max} < \bar{g}_{max}$.

Next, we show that $\max_{\bar{g}}\phi^H(q^*(\bar{g})) > \max_{\bar{g}}\phi(h^{*s}(\bar{g}))$ when $\widehat{h}$ is sufficiently large. Note that the $\bar{g}^{thres}$ and $\bar{g}^{thres,H}$ are defined as $\bar{g}^* = \bar{g}^{thres}F(h^{*s}(\bar{g}^{thres}))$ and $\bar{g}^* = \bar{g}^{thres,H}q(\bar{g}^{thres,H})$, respectively. Using the definition of $\bar{g}^*$ in Assumption 3, we obtain

$$\Pi(h^{*s}(\bar{g}^{thres});\bar{g}^{thres}) = \Psi^{1-\alpha}\alpha(1-\alpha)LF(h^{*s}(\bar{g}^{thres}))\cdot\left(\frac{\theta\xi}{\alpha-\theta}\right)^{\frac{\theta}{\alpha}}, \tag{H.1}$$

$$\Pi^H(q^*(\bar{g}^{thres,H});\bar{g}^{thres,H}) = \Psi^{1-\alpha}\alpha(1-\alpha)Lq^*(\bar{g}^{thres,H})\cdot\left(\frac{\theta\xi}{\alpha-\theta}\right)^{\frac{\theta}{\alpha}}. \tag{H.2}$$

Since

$$\phi(h^{*s}(\bar{g}^{thres})) = \delta L\int_{h^{*s}(\bar{g}^{thres})}^{h_{max}} hdF(h) > \delta Lh^{*s}(\bar{g}^{thres})\cdot\left\{1-F(h^{*s}(\bar{g}^{thres}))\right\},$$

holds, (20) and (21) imply

$$h^{*s}(\bar{g}^{thres}) > \frac{\rho+\sigma\delta Lh^{*s}(\bar{g}^{thres})\cdot\left\{1-F(h^{*s}(\bar{g}^{thres}))\right\}}{\left\{\alpha(1-\alpha)\cdot\left(1-\frac{\xi}{\Pi(h^{*s}(\bar{g}^{thres});\bar{g}^{thres})}\right)-\bar{g}^{thres}\right\}\cdot\frac{LF(h^{*s}(\bar{g}^{thres}))\delta}{\alpha}}.$$

Rearranging the above equation with $\bar{g}^{thres} = \bar{g}^*/F(h^{*s}(\bar{g}^{thres}))$, we have

$$F(h^{*s}(\bar{g}^{thres})) > \frac{1}{\alpha\left\{(1-\alpha)+\sigma\right\}} \cdot \left[\frac{\alpha\rho}{L\delta h^{*s}(\bar{g}^{thres})} + \left(1+\frac{\alpha-\theta}{\theta}\right)\cdot \bar{g}^* + \alpha\sigma\right]. \tag{H.3}$$

Next, let us consider the case of a homogeneous-ability economy. Equations (24) and (25) imply

$$\widehat{h} = \frac{\rho + \sigma\delta L\widehat{h}\cdot(1-q^*(\bar{g}^{thres,H}))}{\left\{\alpha(1-\alpha)\cdot\left(1-\frac{\xi}{\Pi^H(q^*(\bar{g}^{thres,H});\bar{g}^{thres,H})}\right) - \bar{g}^{thres,H}\right\}\cdot\frac{Lq^*(\bar{g}^{thres,H}))\delta}{\alpha}}.$$

Rearranging the above equation with $\bar{g}^{thres,H} = \bar{g}^*/q^*(\bar{g}^{thres,H})$, we obtain

$$q^*(\bar{g}^{thres,H}) = \frac{1}{\alpha\left\{(1-\alpha)+\sigma\right\}} \cdot \left[\frac{\alpha\rho}{L\delta\widehat{h}} + \left(1+\frac{\alpha-\theta}{\theta}\right)\cdot \bar{g}^* + \alpha\sigma\right]. \tag{H.4}$$

Therefore, from (H.3) and (H.4), $F(h^{*s}(\bar{g}^{thres})) > q^*(\bar{g}^{thres,H})$ holds when $\widehat{h}$ is sufficiently large.

Finally, we show that $\max_{\bar{g}} \phi^H(q^*(\bar{g})) > \max_{\bar{g}} \phi(h^{*s}(\bar{g}))$. From the definitions of $\bar{g}^{thres,H}$ and $\bar{g}^{thres}$, the following equations hold:

$$\begin{aligned}
\max_{\bar{g}} \phi^H(q^*(\bar{g})) &= \phi^H(q^*(\bar{g}^{thres,H})) = \delta L\widehat{h}\cdot(1-q^*(\bar{g}^{thres,H})),\\
\max_{\bar{g}} \phi(h^{*s}(\bar{g})) &= \phi(h^{*s}(\bar{g}^{thres})) = \delta L\int_{h^{*s}(\bar{g}^{thres})}^{h_{max}} hdF(h) < \delta L h_{max}\cdot(1-F(h^{*s}(\bar{g}^{thres}))).
\end{aligned}$$

Since $F(h^{*s}(\bar{g}^{thres})) > q^*(\bar{g}^{thres,H})$ holds when $\widehat{h}$ is sufficiently large, $\max_{\bar{g}} \phi^H(q^*(\bar{g})) > \max_{\bar{g}} \phi(h^{*s}(\bar{g}))$ holds when $\widehat{h}$ is sufficiently large.

**Q.E.D.**

# I Proof of Proposition 4

First, let us show that

$$\left.\frac{d\phi(h^{*s}(\bar{g}))}{d\bar{g}}\right|_{\bar{g}=\bar{g}_{max}} \quad < \quad \frac{d\phi^{H}(q^{*}(\bar{g}))}{d\bar{g}} \leq 0 \; \forall \bar{g} \in [\bar{g}^{thres,H}, \bar{g}^{H}_{max}],$$

holds when $h_{max}$ is sufficiently large. Since $h^{*s}(\bar{g}_{max}) = h_{max}$ holds, (23) is rewritten as

$$\left.\frac{d\phi(h^{*s})}{d\bar{g}}\right|_{\bar{g}=\bar{g}_{max}}$$
$$= \frac{\delta L h_{max} \cdot \left\{ \left( \frac{\bar{g}^{*}}{\bar{g}_{max}} \right)^{\frac{\alpha}{\alpha-\theta}} - 1 \right\}}{\frac{1}{h_{max} F'(h_{max})} \cdot \left[ \alpha(1-\alpha) \left( 1 - \frac{\xi}{\Pi_{lim}(\bar{g}_{max})} \right) - \bar{g}_{max} \right] + \left[ \alpha\sigma + \alpha(1-\alpha) + \bar{g}_{max} \cdot \left\{ \left( \frac{\bar{g}^{*}}{\bar{g}_{max}} \right)^{\frac{\alpha}{\alpha-\theta}} - 1 \right\} \right]}, \tag{I.1}$$

where

$$\Pi_{lim}(\bar{g}_{max}) \equiv \Pi(h^{*s}(\bar{g}_{max}), \bar{g}_{max}) = \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} \alpha(1-\alpha) \bar{g}_{max}^{\frac{\theta}{\alpha-\theta}} L^{\frac{\alpha}{\alpha-\theta}}.$$

From (C.1) and Figure C.2, there exists $\bar{g}^{lim}_{max} \in (0,1)$ such that $\bar{g}_{max} \to \bar{g}^{lim}_{max}$ as $h_{max} \to +\infty$, and $\left\{ \left( \bar{g}^{*}/\bar{g}^{lim}_{max} ) \right)^{\frac{\alpha}{\alpha-\theta}} - 1 \right\} < 0$. When $\lim_{h_{max} \to +\infty} h_{max} F'(h_{max}) \neq 0$, we obtain

$$\lim_{h_{max} \to +\infty} \left.\frac{d\phi(h^{*s}(\bar{g}))}{d\bar{g}}\right|_{\bar{g}=\bar{g}_{max}} = -\infty. \tag{I.2}$$

Since

$$-\infty < \frac{d\phi^{H}(q^{*}(\bar{g}))}{d\bar{g}} \leq 0 \; \forall \bar{g} \in [\bar{g}^{thres,H}, \bar{g}^{H}_{max}],$$

is obvious, we can conclude that when $h_{max}$ is sufficiently large, equation (I.1) holds.

Next, we show that

$$\frac{d\phi(h^{*s}(\bar{g}))}{d\bar{g}}\bigg|_{\bar{g}=\bar{g}_{min}} \quad > \quad \frac{d\phi^{H}(q^{*}(\bar{g}))}{d\bar{g}} \geq 0 \;\forall \bar{g} \in [\bar{g}^{H}_{min}, \bar{g}^{thres,H}], \tag{I.3}$$

holds when $h_{max}$ is sufficiently large. Since $h^{*s}(\bar{g}_{min}) = h_{max}$, (23) is rewritten as

$$\frac{d\phi(h^{*s})}{d\bar{g}}\bigg|_{\bar{g}=\bar{g}_{min}}$$
$$= \frac{\delta L h_{max} \cdot \left\{ \left( \frac{\bar{g}^{*}}{\bar{g}_{min}} \right)^{\frac{\alpha}{\alpha-\theta}} - 1 \right\}}{\frac{1}{h_{max}F'(h_{max})} \cdot \left[ \alpha(1-\alpha)\left(1 - \frac{\xi}{\Pi_{lim}(\bar{g}_{min})}\right) - \bar{g}_{min} \right] + \left[ \alpha\sigma + \alpha(1-\alpha) + \bar{g}_{min} \cdot \left\{ \left( \frac{\bar{g}^{*}}{\bar{g}_{min}} \right)^{\frac{\alpha}{\alpha-\theta}} - 1 \right\} \right]}, \tag{I.4}$$

where

$$\Pi_{lim}(\bar{g}_{min}) \equiv \Pi(h^{*s}(\bar{g}_{min}), \bar{g}_{min}) = \Psi^{\frac{\alpha(1-\alpha)}{\alpha-\theta}} \alpha(1-\alpha) \bar{g}_{min}^{\frac{\theta}{\alpha-\theta}} L^{\frac{\alpha}{\alpha-\theta}}.$$

From (C.1) and Figure C.2, there exists $\bar{g}^{lim}_{min} \in (0,1)$ such that $\bar{g}_{min} \to \bar{g}^{lim}_{min}$ as $h_{max} \to +\infty$, and $\left\{ \left( \bar{g}^{*}/\bar{g}^{lim}_{min} \right) \right)^{\frac{\alpha}{\alpha-\theta}} - 1 \right\} > 0$. When $\lim_{h_{max}\to+\infty} h_{max}F'(h_{max}) \neq 0$, we obtain

$$\lim_{h_{max}\to+\infty} \frac{d\phi(h^{*s}(\bar{g}))}{d\bar{g}}\bigg|_{\bar{g}=\bar{g}_{min}} = +\infty. \tag{I.5}$$

Since

$$0 \leq \frac{d\phi^{H}(q^{*}(\bar{g}))}{d\bar{g}} < +\infty \;\forall \bar{g} \in [\bar{g}^{H}_{min}, \bar{g}^{thres,H}],$$

is obvious, we can conclude that when $h_{max}$ is sufficiently large, (I.3) holds.

**Q.E.D.**

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