Public debt rule breaking by time-inconsistent voters*

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October 29, 2020

Working Paper Series No.55
Faculty of Economics, Doshisha University

Abstract

This study considers how present-biased preferences influence public debt policy when a violation of debt rules is possible. To address this issue, the study extends the framework of Bisin, Lizzeri, and Yariv (American Economic Review 105, (2015), 1711–1737) by allowing for rule breaking with extra costs; we show that rule breaking occurs when a country exhibits a strong present bias. We further extend the model by introducing a political process for determining the debt rule, and we show that a polarization of debt rules emerges between countries with high and low degrees of present bias.

Key words: Debt ceilings; Present bias; Public debt.

JEL Classification: D72, D78, H62, H63

*Financial support from the Japan Society for the Promotion of Science Grants-in-Aid for Scientific Research (C) (No. 17K03620, Ryo Arawatari; No. 18K01650, Tetsuo Ono) is gratefully acknowledged. Declarations of interest: none.

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1 Introduction

In the last decade, many developed countries have experienced large budget deficits and rapidly growing public debt. In 2007, the average general government gross debt in Organization for Economic Co-operation and Development (OECD) member countries, as a percentage of GDP, was 53.34%, and it increased to 85.58% in 2015. In particular, the ratio increased by more than 60 points in Greece, Japan, Portugal, Spain, and the United Kingdom.[1] This raises concerns about the sustainability of public finances and highlights the need for fiscal rules in achieving fiscal consolidation (IMF, 2009).

Fiscal rules are expected to constrain the behavior of governments, but the enforceability of these rules is questionable, as indicated by Alesina and Passalacqua (2016). As reported in Wyplosz (2013), the United Kingdom adopted two fiscal rules in 1997: (1) the budget deficit may only finance public investment and (2) the debt-to-GDP ratio may not exceed 40 percent. However, while the rule was met for a few years, slippage set in after 2002. Wyplosz (2013) also reports that in the euro area, the Maastricht treaty specifies that budget deficits cannot exceed 3 percent, but this rule was satisfied only 60 percent of the time in the first thirteen years of the existence of euro. The evidence suggests that, in practice, the conditions required for fiscal institutions are rarely met.

To investigate why fiscal rule violations occur so frequently, the present study focuses on time-inconsistent, present-biased preferences. When agents are endowed with such preferences, they change their ex-ante consumption plans, choosing to consume more in the present and less in the future (Laibson, 1997). In particular, they are incentivized to support, via voting, a large public debt issue; this enables them to obtain a great deal of resources for consumption today through transfers financed by the debt issuance. Bisin, Lizzeri, and Yariv (2015) is the first study to present a model of public debt that includes such an incentive mechanism.

In the framework of Bisin, Lizzeri, and Yariv (2015), the government, representing the present-biased agents, is assumed to stick to a given debt rule; rule breaking is abstracted away from their analysis. However, if rule breaking is available through the payment of extra costs, the agents may find it optimal to support the issue of public debt that surpasses the debt ceiling. Such rule breaking depends on the degree of present bias, but this degree differs among countries, as reported in Wang, Rieger, and Hens (2016). Thus, the following questions arise naturally: (a) how do present-biased preferences influence the choice of public debt issuance when rule breaking is possible and (b) what kind of a debt rule must be put in place in response to the degree of present bias. The purpose of this study is to address these questions.

[1] Source: OECD.stat. https://stats.oecd.org/ (Accessed on June 12, 2019).

For analysis, we use the simple three-period model developed by Bisin, Lizzeri, and Yariv (2015). Agents are endowed with goods in period 1, and they make savings and portfolio decisions. They then receive utility from consumption in periods 2 and 3. The period-2 selves are endowed with present-biased preferences, so they are tempted to increase consumption in period 2 at the cost of reduced consumption in period 3. The period-1 selves use illiquid assets to constrain the consumption plans of their future selves. However, the government, representing the period-2 selves, is induced to issue public debt to respond to the desire of the period-2 selves to undo the commitment from period 1. This gives sophisticated agents an incentive to rebalance their portfolios in period 1 to reestablish their commitment consumption sequence. This, in turn, creates demand for further debt accumulation.

Bisin, Lizzeri, and Yariv (2015) control the behavior of present-biased agents by imposing a debt ceiling. Our model differs from theirs in that debt issue beyond the ceiling is available by incurring some additional costs. Within this extended framework, we show that the benefits of rule breaking outweigh the costs, and, thus, rule breaking occurs if the present bias is extremely strong and the debt ceiling is fairly low. The result could be viewed as providing one possible key to understanding the phenomenon of fiscal rule breaking often observed in developed countries.

The assumption of a fixed debt ceiling follows that used in Bisin, Lizzeri, and Yariv (2015). This assumption is reasonable in the short run, but in the long run there is, in reality, a tendency toward the revision of fiscal rules. For instance, according to the US Department of the Treasury, the US debt ceiling has been raised 78 times since 1960.[2] Another example involves Japan, which has the highest debt-to-GDP ratio among all developed countries. Despite the urgent need for fiscal consolidation, Japan pushed back the target year for achieving a primary balance surplus from 2020 to 2025 given its slow recovery from the recession.[3] The evidence suggests that fiscal rules are not necessarily rigorous requirements; rather, they can be easily revised in accordance with economic and political conditions.

Motivated by this evidence, we extend the analysis by introducing the endogenous determination of the debt ceiling via voting. In particular, we consider a situation where the period-1 selves set the debt ceiling, taking into account the response of the period-2 selves. Under this setting, we investigate how the present-biased preferences affect the design of the debt ceiling, and we show the following result. When the present bias is weak, the period-2 selves have little incentive to change their consumption plans and thus

[2] Source: https://home.treasury.gov/policy-issues/financial-markets-financial-institutions-and-fiscal-service/debt-limit (accessed on September 27, 2019)

[3] Source: https://www5.cao.go.jp/keizai-shimon/kaigi/cabinet/2018/2018_basicpolicies_en.pdf (accessed on September 27, 2019)

to issue public debt. They follow this debt rule even if the ceiling is set at the lowest level, that is, even if a debt issue is prohibited. Therefore, the debt ceiling is set at zero, which is optimal from the viewpoint of the period-1 selves' utility maximization.

When the present bias is strong, the marginal benefits of rule breaking outweigh the costs for the period-2 selves. Thus, the period-1 selves find it impossible to make the period-2 selves follow the rule of no debt issue. To avoid the costs of rule breaking, the period-1 selves set the ceiling at a maximum level such that the period-2 selves never break it. The result, thus far, suggests that there is a threshold level for the present bias such that the optimal ceiling for the period-1 selves varies in a substantial manner around the threshold. This result could be viewed as providing one possible explanation for the cross-county difference in debt rules and the resulting diverse levels of public debt among developed countries.

As mentioned above, the present study is closely related to Bisin, Lizzeri, and Yariv (2015), who demonstrate the role of present-biased preferences in fiscal policy making. The study is also related to Halac and Yared (2018), who analyze the formation of fiscal rules in the presence of present-biased preferences in a multi-country economy. In particular, they compare coordinated rules, chosen jointly by a group of countries, to uncoordinated rules, chosen independently by each country. They show that the coordinated rules are slacker when the present bias is large. However, rule breaking is abstracted away from their analysis. The present study, in contrast, allows rule breaking, and it derives the optimal rules under the possibility of rule breaking by present-biased voters.

The present study also contributes to the literature on the political economy of public debt, such as Cukierman and Meltzer (1989); Song, Storesletten, and Zilibotti (2012); Azzimonti, Battaglini, and Coate (2016); Barseghyan and Battaglini (2016); and Arai, Naito, and Ono (2018). In all of these studies, it is assumed that agents are not present biased, and fiscal rules are taken as given. The present work advances the literature by relaxing these assumptions, and it shows how public debt accumulation and the determination of fiscal rules are affected by the present-biased preferences of voters.

This paper is organized as follows: The next section lays out the model. The third section demonstrates the agents' saving decisions and the government's fiscal policy decisions. The fourth section characterizes the equilibrium allocation. The fifth section extends the model to the endogenous determination of debt rules, and the last section concludes. Proofs for the propositions are in the appendix.

2 The Model

The model is based on the one developed by Bisin, Lizzeri, and Yariv (2015). It measures identical agents who live for three periods: 1, 2, and 3. They are endowed with k units of goods in period 1 and nothing in periods 2 and 3. In period 1, agents only make savings and portfolio decisions; they receive utility from consumption in periods 2 and 3.

Agents (hereafter interchangeably called individuals, selves, and voters) have time-inconsistent, present-biased preferences (Laibson, 1997). In particular, the agents' preferences regarding consumption in periods 2 and 3, c_2 and c_3, are given by the following utility functions:

$$U_1(c_2, c_3) = \beta \left[u(c_2) + u(c_3)\right],$$
$$U_2(c_2, c_3) = u(c_2) + \beta u(c_3),$$

where U_t $(t = 1, 2)$ is the assessed utility at time t, u is a continuous and strictly concave utility function, and $\beta \in (0, 1)$ is a parameter representing the degree of present bias; a lower β implies that the period-2 agents are biased toward more period-2 consumption. Agents are assumed to be sophisticated; they are fully aware of their self-control problems.

Agents choose to invest their wealth, k, in liquid or illiquid assets in period 1. It is assumed that all liquid and illiquid assets have the same exogenous interest rate of zero. Liquid assets are one-period securities that are sold in period t $(t = 1, 2)$ and redeemed in period $t + 1$. Illiquid assets are two-period securities that are sold in period 1 and redeemed in period 3; they cannot be sold in period 2. Savings in one- and two-period securities in period 1 are denoted by, s_{12} and s_{13}, respectively; the subscript ij means the time of saving, i, and redemption, j. In period 2, agents can save the return from s_{12} in one-period securities; this saving is denoted by s_{23}.

Agents displaying present-biased preferences suffer from self-control problems. In particular, the period-2 selves are tempted to increase consumption in period 2 at the cost of reduced consumption in period 3. The period-1 selves use illiquid assets to constrain the consumption plans of their future selves. However, the government, representing the period-2 selves, is induced to issue public debt in the international market to respond to the period-2 selves' desire to undo the commitment made in period 1. This gives sophisticated agents an incentive to rebalance their portfolios in period 1 to reestablish their consumption sequence commitment. This, in turn, creates demand for further debt accumulation. The debt issue, denoted by d, is assumed to be costly and constrained by the constitutionally imposed borrowing limits denoted by $\bar{d}$, but debt issue beyond the limit is available by incurring some additional costs, as specified below.

The budget constraints in periods 1, 2, and 3 are given by

$$\text{period 1: } s_{12} + s_{13} \leq k,$$
$$\text{period 2: } c_2 \leq s_{12} + d - s_{23},$$
$$\text{period 3: } c_3 \leq s_{13} + s_{23} - G(d),$$

where $G(d)$ represents the costs of debt repayment, specified as follows:

$$G(d) = \left[\begin{array}{cc} (1+\eta)d & \text{when } d \leq \bar{d} \\ (1+\eta)d + \gamma(d-\bar{d}) & \text{when } d > \bar{d}, \end{array} \right.$$

where $\eta > 0$ and $\gamma > 0$. Debt is financed by foreign lenders at an interest rate of zero, but it can be directly distortionary. The term η represents the marginal cost of debt issue, such as labor supply distortions induced by increased tax burdens for debt repayment (Bisin, Lizzeri, and Yariv, 2015). The term γ, introduced in this study, represents the marginal costs of issuing public debt, conditional on the level of debt being above $\bar{d}$. Such costs could be viewed as reputational losses for rule-breaking countries (Eyraud et al., 2020).

The timing of events and optimization problem at each stage are as follows. In period 1, an agent, who predicts an equilibrium per capita public debt level of d, chooses period 1 savings intended for period 2, s_{12}, and period 3, s_{13}, to maximize the assessed utility in period 1, U_1. Since the debt level is determined by the government, representing the period-2 selves, each agent takes it as given when making his or her saving decision. The problem of the agent in period 1 is

$$\begin{array}{ll} \max\limits_{s_{12},s_{13}} & \beta\left[u\left(s_{12}+d^e-s_{23}\left(s_{12}\right)\right)+u\left(s_{13}+s_{23}\left(s_{12}\right)-G(d^e)\right)\right], \\ \text{s.t.} & s_{12}+s_{13}\leq k, \\ & s_{12}\geq 0, s_{13}\geq 0, \\ & \text{given } d^e, \end{array}$$

where d^e denotes the expected level of public debt issue in period 2, which is taken as given. We assume rational expectations of the equilibrium level of debt issue, which will be defined in the next section. The term $s_{23}(s_{12})$ implies that agents know that their choice of s_{12} (and thus s_{13}) will have an effect on the period-2 saving choice, s_{23}. Private borrowing is not allowed, following Bisin, Lizzeri, and Yariv (2015).

In period 2, an agent chooses the savings intended for period 3, s_{23}, to maximize the assessed utility in period 2, taking the expected level of per capita public debt, d^e, as given. The problem of the period-2 agent is

$$\begin{array}{ll} \max\limits_{s_{23}} & u\left(s_{12}+d^e-s_{23}\right)+\beta u\left(s_{13}+s_{23}-G(d^e)\right), \\ \text{s.t.} & s_{23}\geq 0 \\ & \text{given } s_{12}, s_{13}, \text{ and } d^e. \end{array}$$

We assume that period-1 and -2 agents have the same expectations for the debt, d^e.

The government, representing the period-2 selves, chooses public debt issue, d, to maximize the utility of the period-2 agents, subject to a non-negativity constraint, $d \geq 0$, and a constitutionally imposed debt ceiling, $\bar{d}$, given s_{12}, s_{13}, and s_{23}.[4]

$$\begin{aligned} \max_{d} \quad & u\left(s_{12} + d - s_{23}\right) + \beta u\left(s_{13} + s_{23} - G(d)\right), \\ \text{s.t.} \quad & d \geq 0, \\ & \text{given } s_{12}, s_{13}, \text{and } s_{23}. \end{aligned}$$

Figure 1 illustrates the timing of events.

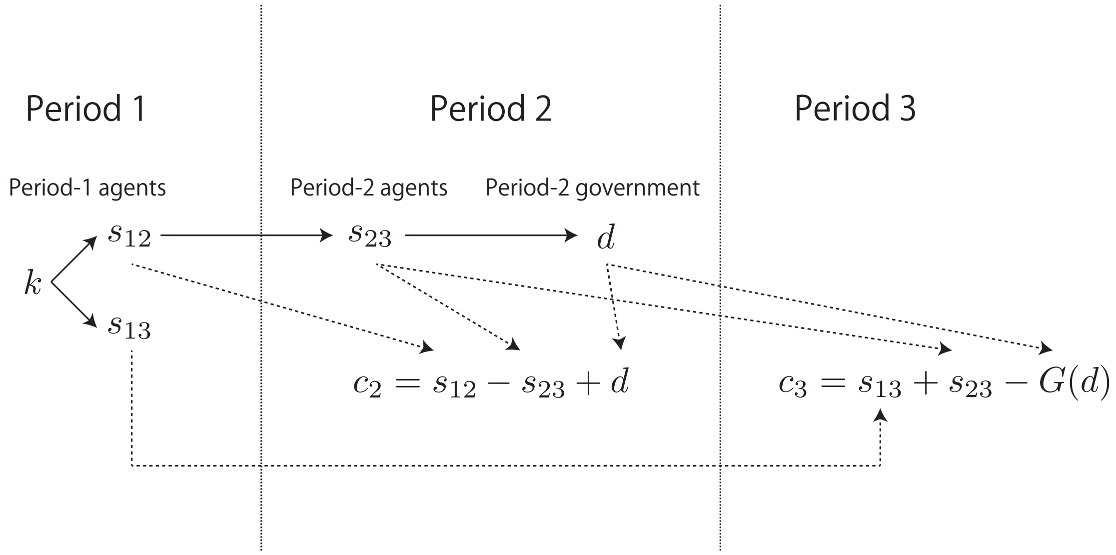

Figure 1: Timing of events.

Two remarks are in order. First, in the present framework, the government represents agents' preferences perfectly since agents are assumed to be identical. This perfect representation makes the difference between agents and the government unclear at first glance. A key to the difference between them is that agents choose their saving taking future public debt issue, d, as given, while the government, as a collective entity, can control public debt issue. This implies that no one can deviate from the collective decision making on public debt. Second, related to the first point, d represents public debt, not private debt. If d is the private debt, then period-1 selves can control the level of d via the choice of saving in period 1 (Bissin, Lizzeri, and Yariv, 2015). Such a case is not considered in the present study.

For our analysis, we make the following assumptions. First, the utility function is specified as

$$u\left(c\right) = \frac{\left(c\right)^{1-\sigma} - 1}{1 - \sigma},$$

where σ (> 0) is an inverse of the inter-temporal elasticity of substitution. This assumption enables us to solve the model analytically, but has no substantial effect on the following

[4]Lending in the international market, $d < 0$, is abstracted away from the analysis since our focus is on borrowing, $d > 0$. Allowing for $d < 0$ does not qualitatively alter the following result.

analysis. In particular, the results shown below hold true regardless of whether σ is greater or less than one. Second, the borrowing must be below the natural debt limit, k/η, to prevent the government from defaulting. In addition, to define the debt ceiling, it is assumed that the ceiling is below the natural debt limit, as in the following assumption:

Assumption 1 $\bar{d} < k/\eta$.[5]

3 Decisions of Agents and Government

As mentioned above, agents are assumed to be sophisticated. Thus, we solve the model through backward induction; that is, we first solve the government's problem in period 2, then the agents' problem in period 2, and finally the agents' problem in period 1. Our result would not change if the timing within period 2 is reversed because the period-2 selves and the government share the same objective.

3.1 Government's Period-2 Decision

The objective of the government, representing the period-2 selves, is

$$V_g\left(s_{12}, s_{23}, d\right) \equiv \frac{\left(s_{12}+d-s_{23}\right)^{1-\sigma}-1}{1-\sigma}+\beta \cdot \frac{\left(k-s_{12}+s_{23}-G(d)\right)^{1-\sigma}-1}{1-\sigma}.$$

Assuming interior solutions, we can write the first-order conditions with respect to d, when $d \leq \bar{d}$ and $d > \bar{d}$, as follows:

$$\begin{aligned} \left.\frac{\partial V_g\left(s_{12}, s_{23}, d\right)}{\partial d}\right|_{d \leq \bar{d}} &= \left(s_{12}+d-s_{23}\right)^{-\sigma} \\ &\quad -\beta\left(1+\eta\right) \cdot \left(k-s_{12}+s_{23}-(1+\eta)d\right)^{-\sigma} \leq 0, \qquad (1) \\ \left.\frac{\partial V_g\left(s_{12}, s_{23}, d\right)}{\partial d}\right|_{d > \bar{d}} &= \left(s_{12}+d-s_{23}\right)^{-\sigma} \\ &\quad -\beta\left(1+\eta+\gamma\right) \cdot \left(k-s_{12}+s_{23}-(1+\eta+\gamma)d+\gamma\bar{d}\right)^{-\sigma} \leq 0. \qquad (2) \end{aligned}$$

Let d^u and d^c denote interior solutions satisfying the first-order conditions in Eqs. (1) and (2), respectively.[6] They are expressed as functions of s_{12} and s_{23} as follows:

$$d^u\left(s_{12}, s_{23}\right) = \frac{k-\left[1+\left\{\beta\left(1+\eta\right)\right\}^{1/\sigma}\right]\left(s_{12}-s_{23}\right)}{\left(1+\eta\right)+\left\{\beta\left(1+\eta\right)\right\}^{1/\sigma}}, \qquad (3)$$

$$d^c\left(s_{12}, s_{23}\right) = \frac{k+\gamma\bar{d}-\left[1+\left\{\beta\left(1+\eta+\gamma\right)\right\}^{1/\sigma}\right]\left(s_{12}-s_{23}\right)}{\left(1+\eta+\gamma\right)+\left\{\beta\left(1+\eta+\gamma\right)\right\}^{1/\sigma}}, \qquad (4)$$

[5] This assumption works when we solve the period-1 agents' optimization problem, as presented in Appendix A.1.

[6] The superscript "u" and "c" mean that the choice of debt is "unconstrained" and "constrained" by the debt ceiling, respectively. When the choice is constrained, the government can break it by incurring some additional costs.

where $d^u(s_{12}, s_{23})$ and $d^c(s_{12}, s_{23})$ satisfy

$$d^c(s_{12}, s_{23}) \gtreqless \bar{d} \Leftrightarrow A(s_{12}, s_{23}) \equiv \frac{k - \left[1 + \{\beta(1+\eta+\gamma)\}^{1/\sigma}\right](s_{12} - s_{23})}{(1+\eta) + \{\beta(1+\eta+\gamma)\}^{1/\sigma}} \gtreqless \bar{d},$$

and[7]

$$A(s_{12}, s_{23}) \leq d^u(s_{12}, s_{23}).$$

The condition of $A(s_{12}, s_{23}) \leq d^u(s_{12}, s_{23})$ implies that there are four possible cases, classified according to the relative magnitude among $d^u(s_{12}, s_{23})$, $A(s_{12}, s_{23})$, and $\bar{d}$, as illustrated in Figure 2: $d^u(s_{12}, s_{23}) \leq 0 \leq \bar{d}$ (Panel (a)), $0 < d^u(s_{12}, s_{23}) < \bar{d}$ (Panel (b)), $A(s_{12}, s_{23}) \leq \bar{d} \leq d^u(s_{12}, s_{23})$ (Panel (c)), and $\bar{d} < A(s_{12}, s_{23})$ (Panel (d)). From the figure, we can find the solution, d, for the government's optimization problem, denoted by $d^*(s_{12}, s_{23})$, as follows:

$$d^*(s_{12}, s_{23}) = \begin{cases} 0 & \text{when } d^u(s_{12}, s_{23}) \leq 0, \\ d^u(s_{12}, s_{23}) & \text{when } 0 < d^u(s_{12}, s_{23}) < \bar{d}, \\ \bar{d} & \text{when } A(s_{12}, s_{23}) \leq \bar{d} \leq d^u(s_{12}, s_{23}), \\ d^c(s_{12}, s_{23}) & \text{when } \bar{d} < A(s_{12}, s_{23}). \end{cases} \tag{5}$$

Consider first $d^u(s_{12}, s_{23})$, which represents the optimal level of public debt when it satisfies the debt ceiling. Eq. (3) indicates that $d^u(s_{12}, s_{23})$ increases as $(s_{12} - s_{23})$ and β decrease. The term $(s_{12} - s_{23})$, representing the period-2 consumption when there is no debt issue, implies that the marginal utility of the period-2 consumption increases as $(s_{12} - s_{23})$ decreases. The term β, representing the present bias, implies that the period-2 agents attach a larger weight to the period-2 consumption relative to the period-3 consumption as β decreases. Thus, the period-2 selves' preferences for debt financing increase as $(s_{12} - s_{23})$ and β decrease.

More precisely, suppose first that $(s_{12} - s_{23})$ and β are high, such that $d^u(s_{12}, s_{23}) \leq 0$ holds. Then the optimal level of the public debt is below zero. In other words, the government prefers to lend rather than borrow in the international market. However, lending is not allowed in the present framework. Thus, the government's choice is constrained by the non-negativity constraint; the optimal level of public debt becomes $d^* = 0$, as illustrated in Panel (a) of Figure 2. When $(s_{12} - s_{23})$ and β are at moderate levels, such that $0 < d^u(s_{12}, s_{23}) < \bar{d}$, the government is not constrained by the non-negativity constraint or the debt ceiling. Thus, its choice is $d^* = d^u(s_{12}, s_{23})$, as illustrated in Panel (b) of Figure 2.

[7] Proof of $A(s_{12}, s_{23}) \leq d^u(s_{12}, s_{23})$ is as follows. Suppose, to the contrary, that $A(s_{12}, s_{23}) > d^u(s_{12}, s_{23})$, that is, $0 > k/\eta + s_{12} - s_{23}$ holds. The period-2 budget constraint leads to $c_2 \leq s_{12} + d - s_{23} < s_{12} + k/\eta - s_{23}$, where the second inequality comes from $d \leq \bar{d} < k/\eta$. Given $c_2 > 0$, this implies that $0 < s_{12} + k/\eta - s_{23}$, which is a contradiction.

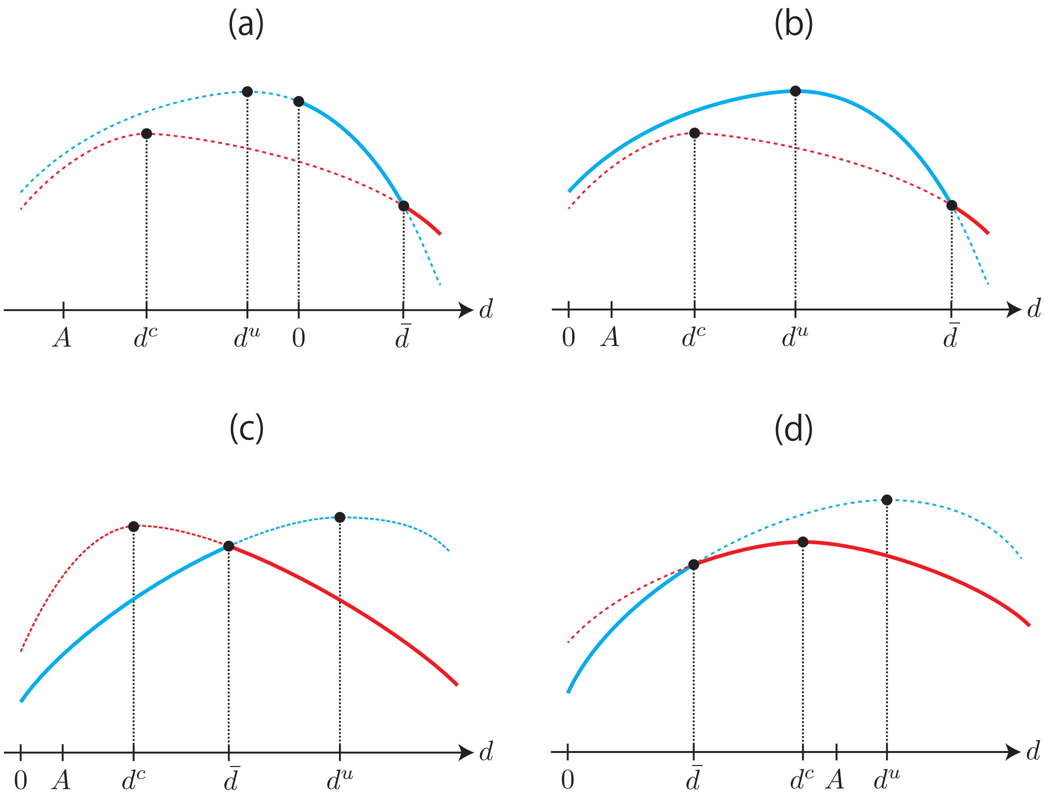

Figure 2: Illustration of the period-2 government's objective function when $d^u(s_{12}, s_{23}) \leq 0 \leq \bar{d}$ (Panel (a)), $0 < d^u(s_{12}, s_{23}) < \bar{d}$ (Panel (b)), $A(s_{12}, s_{23}) \leq \bar{d} \leq d^u(s_{12}, s_{23})$ (Panel (c)), and $\bar{d} < A(s_{12}, s_{23})$ (Panel (d)).

Finally, when $(s_{12} - s_{23})$ and β are low, such that $\bar{d} \leq d^u(s_{12}, s_{23})$ holds, the government may borrow over the debt ceiling. In particular, its decision depends on the relative magnitude between $A(s_{12}, s_{23})$ and $\bar{d}$. Since $A(s_{12}, s_{23})$ is decreasing in γ, which represents the costs of rule breaking, the government finds it is optimal to follow the rule and issues debt up to the limit, $d^* = \bar{d}$, when γ is large, such that $A(s_{12}, s_{23}) \leq \bar{d}$, as illustrated in Panel (c) of Figure 2. However, rule breaking occurs when γ is low, such that $\bar{d} < A(s_{12}, s_{23})$, as illustrated in Panel (d) of Figure 2.

3.2 Agents' Period-2 Decision

Next, we consider the period-2 agents' decision regarding one-period securities, s_{23}. The objective function of the period-2 agents is:

$$V_2(s_{12}, s_{23}, d^e) \equiv \frac{(s_{12} + d^e - s_{23})^{1-\sigma} - 1}{1-\sigma} + \beta \cdot \frac{[k - s_{12} + s_{23} - G(d^e)]^{1-\sigma} - 1}{1-\sigma}.$$

Notice that period-2 agents take d^e as given when choosing s_{23} because they are infinitesimal and thus are unable to control d by choosing s_{23}. The first-order condition with respect to s_{23} leads to:

$$s_{23} = s_{23}^{u}\left(s_{12}, d^{e}\right) \equiv s_{12} - \frac{k - \left[(1+\eta) + (\beta)^{1/\sigma}\right] d^{e}}{1 + (\beta)^{1/\sigma}} \text{ when } d^{e} \leq \bar{d}, \tag{6}$$

$$s_{23} = s_{23}^{c}\left(s_{12}, d^{e}\right) \equiv s_{12} - \frac{k + \gamma\bar{d} - \left[(1+\eta+\gamma) + (\beta)^{1/\sigma}\right] d^{e}}{1 + (\beta)^{1/\sigma}} \text{ when } d^{e} > \bar{d}. \tag{7}$$

With the private borrowing constraint, $s_{23} \geq 0$, and the expectation of $d = d^{e}$, an optimal level of s_{23}, denoted by s_{23}^{*}, is given by

$$s_{23}^{*}(s_{12}, d^{e}) = \begin{cases} 0 & \text{when } d^{e} \leq \bar{d} \text{ and } s_{12} \leq S^{u}\left(d^{e}\right), \\ s_{23}^{u}\left(s_{12}, d^{e}\right) & \text{when } d^{e} \leq \bar{d} \text{ and } s_{12} > S^{u}\left(d^{e}\right), \\ 0 & \text{when } d^{e} > \bar{d} \text{ and } s_{12} \leq S^{c}\left(d^{e}\right), \\ s_{23}^{c}\left(s_{12}, d^{e}\right) & \text{when } d^{e} > \bar{d} \text{ and } s_{12} > S^{c}\left(d^{e}\right), \end{cases} \tag{8}$$

where $S^{u}\left(d^{e}\right)$ and $S^{c}\left(d^{e}\right)$ are defined as follows:

$$S^{u}\left(d^{e}\right) \equiv \frac{k - \left[(1+\eta) + (\beta)^{1/\sigma}\right] d^{e}}{1 + (\beta)^{1/\sigma}}, \tag{9}$$

$$S^{c}\left(d^{e}\right) \equiv \frac{k + \gamma\bar{d} - \left[(1+\eta+\gamma) + (\beta)^{1/\sigma}\right] d^{e}}{1 + (\beta)^{1/\sigma}}. \tag{10}$$

The period-2 selves attach a larger weight to period-2 consumption than the period-1 selves. This implies that the former selves are induced to increase their period-2 consumption by lowering their saving in s_{23}. In particular, the period-2 selves find it optimal to save nothing in s_{23} when the expectation of d^{e} is low and/or when the saving in one-period securities, s_{12}, by the period-1 selves is low, such that either $s_{12} \leq S^{u}\left(d^{e}\right)$ or $s_{12} \leq S^{c}\left(d^{e}\right)$ holds. If this were not the case, the period-2 selves could afford to save a portion of the return from one-period securities, s_{12}, in s_{23}.

3.3 Agents' Period-1 Decision

Consider the period-1 agents' objective function, which is given by:

$$V_{1}\left(s_{12}, d^{e}\right) \equiv \frac{\left[s_{12} + d^{e} - s_{23}^{*}(s_{12}, d^{e})\right]^{1-\sigma} - 1}{1-\sigma} + \frac{\left[k - s_{12} + s_{23}^{*}(s_{12}, d^{e}) - G\left(d^{e}\right)\right]^{1-\sigma} - 1}{1-\sigma}.$$

Given the expectation of $d = d^{e}$, the period-1 agents choose s_{12} to maximize their objective. Let s_{12}^{*} denote the solution to the problem. The solution satisfies the following first-order condition:

$$\frac{\partial V_{1}\left(s_{12}, d^{e}\right)}{\partial s_{12}} = \left[1 - \frac{\partial s_{23}^{*}(s_{12}, d^{e})}{\partial s_{12}}\right] \cdot \left(s_{12} + d^{e} - s_{23}^{*}(s_{12}, d^{e})\right)^{-\sigma} - \left[1 - \frac{\partial s_{23}^{*}(s_{12}, d^{e})}{\partial s_{12}}\right] \cdot \left[k - s_{12} + s_{23}^{*}(s_{12}, d^{e}) - G\left(d^{e}\right)\right]^{-\sigma} \leq 0, \tag{11}$$

where a strict inequality holds if $s_{12} = 0$.

4 Equilibrium

Having described the behavior of agents and the government, we define an equilibrium in the present framework as follows.

Definition 1: A *rational expectations equilibrium* is an allocation $(s_{12}, s_{13}, s_{23}, c_2, c_3, d)$, such that (i) $s_{12} = s_{12}^*(d^e)$ solves the period-1 agents' problem given s_{23} and $d = d^e$; (ii) $s_{23} = s_{23}^*(s_{12}^*(d^e), d^e)$ solves the period-2 agents' problem given s_{12} and $d = d^e$; (iii) rational expectations hold, that is, the solution to the period-2 government's problem, d, satisfies $d^*\left(s_{12}^*(d^e), s_{23}^*(s_{12}^*(d^e), d^e)\right) = d^e$; and (iv) given $s_{12} = s_{12}^*(d^e)$, $s_{23} = s_{23}^*(s_{12}^*(d^e), d^e)$, and $d = d^*\left(s_{12}^*(d^e), s_{23}^*(s_{12}^*(d^e), d^e)\right)$, the allocation (s_{13}, c_2, c_3) is determined by the period-1, -2, and -3 budget constraints.

Figure 3 is an illustration of the rational expectations equilibrium.

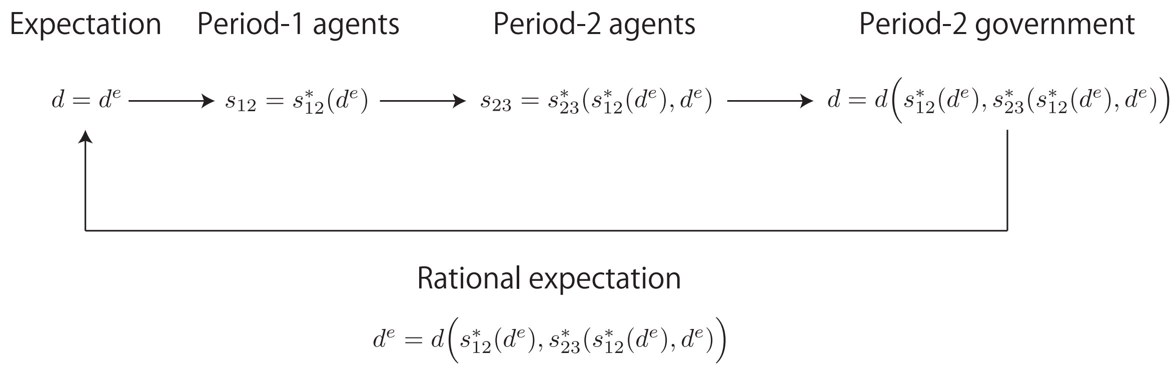

Figure 3: Rational expectations equilibrium.

To characterize the equilibrium allocation, we proceed with the analysis in the following manner. First, we assume that the period-1 and -2 selves have one of the following expectations of d: (i) $d^e = 0$, (ii) $d^e = d^u \in \left(0, \bar{d}\right)$, (iii) $d^e = \bar{d}$, or (iv) $d^e = d^c (> \bar{d})$, where d^u and d^c denote the expectations of agents that the debt issuance is below or above the ceiling, $\bar{d}$, respectively. Given the expectation of the debt issuance, we solve for one-period securities, $s_{12} = s_{12}^*(d^e)$ and $s_{23} = s_{23}^*(s_{12}^*(d^e), d^e)$. Then, we substitute these into the solution, $d = d^*\left(s_{12}, s_{23}\right)$, for the government problem and identify the condition in which the expectations are rational.

Let $\bar{d}^L(\beta)$ and $\bar{d}^H(\beta)(> \bar{d}^L(\beta))$ denote the two threshold values of public debt:

$$\bar{d}_L(\beta) \equiv \frac{k}{(1+\eta) + \{\beta(1+\eta+\gamma)\}^{\frac{1}{\sigma}}}, \tag{12}$$

$$\bar{d}_H(\beta) \equiv \frac{k}{(1+\eta) + \{\beta(1+\eta)\}^{\frac{1}{\sigma}}}. \tag{13}$$

With the use of these two threshold values, we can present the equilibrium level of public debt in the following proposition. The corresponding allocation of saving and consumption is presented in the appendix.

Proposition 1 (Equilibrium Public Debt)

(i) *If* $1/(1+\eta) < \beta$ *holds, then the equilibrium debt is below the ceiling,* $d^* < \bar{d}$.

(ii) *If* $\beta = 1/(1+\eta)$ *holds, then there is a continuum of equilibrium levels of debt, (a)* $d^* \in [0, \bar{d}]$, *if* $\bar{d} < k/(2+\eta)$, *and (b)* $d^* \in [0, k/(2+\eta)]$, *otherwise.*

(iii) *If* $1/(1+\eta+\gamma) < \beta < 1/(1+\eta)$ *holds, then the equilibrium debt is (a) below the ceiling,* $d^* < \bar{d}$, *if* $\bar{d}_H(\beta) < \bar{d}$, *and (b) up to the ceiling,* $d^* = \bar{d}$, *otherwise.*

(iv) *If* $\beta = 1/(1+\eta+\gamma)$ *holds, (a) there is a continuum of equilibrium levels of debt,* $d^* \in [\bar{d}, (k+\gamma\bar{d})/(2+\eta+\gamma))]$, *if* $\bar{d} < k/(2+\eta)$; *(b) the equilibrium debt is up to the ceiling,* $d^* = \bar{d}$, *if* $k/(2+\eta) \leq \bar{d} \leq \bar{d}_H(1/1(1+\eta+\gamma))$; *and (c) the equilibrium debt is below the ceiling,* $d^* < \bar{d}$, *otherwise.*

(v) *If* $\beta < 1/(1+\eta+\gamma)$ *holds, then the equilibrium debt is (a) below the ceiling,* $d^* < \bar{d}$, *if* $\bar{d}_H(\beta) < \bar{d}$; *(b) up to the ceiling,* $d^* = \bar{d}$, *if* $\bar{d}_L(\beta) \leq \bar{d} \leq \bar{d}_H(\beta)$; *and (c) beyond the ceiling,* $d^* > \bar{d}$, *otherwise.*

Proof. See Appendix A.1.

Figure 4 takes β in the horizontal axis and $\bar{d}$ in the vertical axis, and it illustrates the classification of equilibrium states according to the level of public debt. Figure 5 illustrates how the equilibrium levels of public debt, d^*, and the one-period securities from period 1 to period 2, s^*_{12}, change in response to a change in the degree of present bias, represented by β. As observed from Figure 5, a lower β is associated with a higher level of d^* and a lower level of s^*_{12}. A lower β implies that period-2 agents are more present biased relative period-1 agents. In other words, the period-2 consumption, as planned by the period-2 agents, is excessive from the period-1 agents' viewpoint. To establish control over period-2 consumption, period-1 agents reduce the one-period securities, s^*_{12}, that contribute to period-2 consumption, and instead increase two-period securities, s^*_{13}. Given this behavior of period-1 agents, period-2 agents, as voters, support more public debt issue to increase their consumption in period 2.

The result in Proposition 1 shows that the equilibrium level of public debt substantially changes around the two threshold values of β, $1/(1+\eta)$ and $1/(1+\eta+\gamma)$. The mechanism behind these changes is as follows. First, consider the choice of public debt around $\beta = 1/(1+\eta)$. When β is slightly higher than the threshold, $1/(1+\eta)$, period-2 agents have little incentive to support public debt issue via voting. Period-1 agents can control

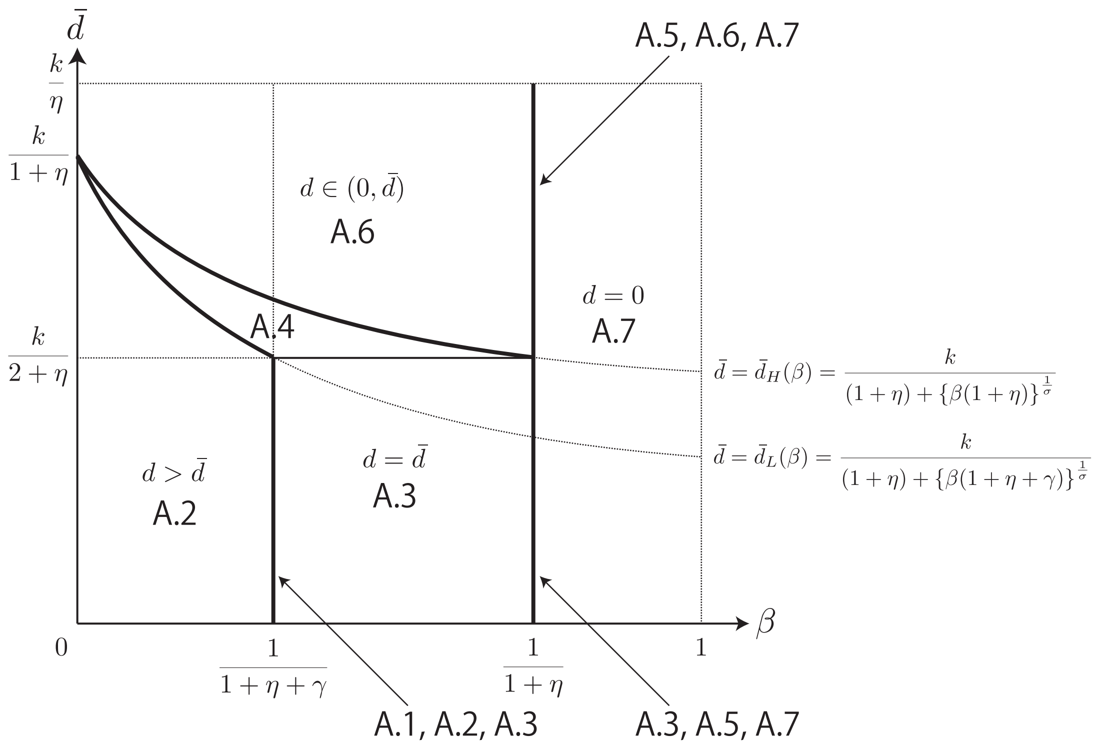

Figure 4: Classification of the equilibrium states according to the level of public debt. The horizontal axis takes β; the vertical axis takes $\bar{d}$. The symbols from A.1 to A.7 correspond to the proposition numbers in the appendix.

period-2 agents' decisions through saving decisions in period 1 and attain the first-best allocation. However, when β is slightly lower than the threshold, $1/(1+\eta)$, period-1 agents are unable to prevent period-2 agents from issuing public debt. This limitation induces period-1 agents to cut the savings on one-period securities, s_{12}^*, substantially. To compensate for this loss of savings, period-2 agents choose to increase the public debt issue considerably. This is the mechanism behind the substantial changes in the amount of public debt around the threshold $\beta = 1/(1+\eta)$.

Second, consider the choice of public debt around $\beta = 1/(1+\eta+\gamma)$. When β is slightly higher than the threshold, $1/(1+\eta+\gamma)$, period-2 agents support and choose public debt issue up to the ceiling, $d^* = \bar{d}$. This implies that for a β slightly lower than the threshold, $1/(1+\eta+\gamma)$, period-2 agents have an incentive to support the public debt issue beyond the ceiling at the cost of rule breaking. Given this expected behavior on the part of the period-2 agents, the period-1 agents reduce the one-period securities, s_{12}^*, to control the excess consumption in period 2. The period-2 agents, in turn, choose to increase public debt issue to compensate for the loss of the return from one-period securities. Therefore, there is a substantial change in the amount of public debt around the threshold of $\beta = 1/(1+\eta+\gamma)$, as illustrated in Panel (a) of Figure 5. However, such change does not occur in the cases shown in Panel (b) and Panel (c) because the debt

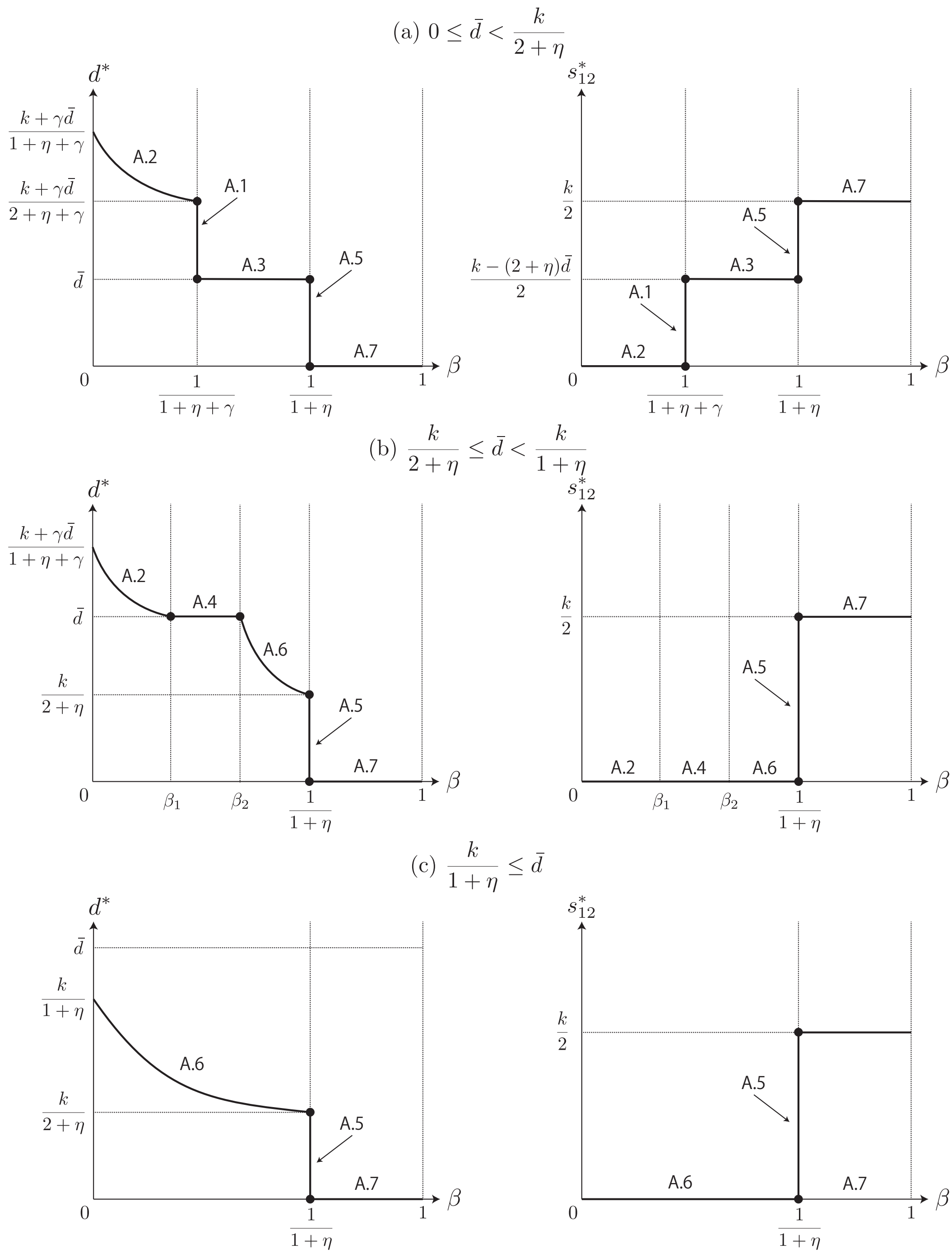

Figure 5: The figure illustrates how the equilibrium debt (d^*) and the one-period securities from period 1 to period 2 (s^*_{12}) change in response to a change in β. The parameters β_1 and β_2 shown in Panel (b) are defined as $\beta_1 \equiv \frac{1}{1+\eta+\gamma}\left(\frac{k-(1+\eta)\bar{d}}{\bar{d}}\right)^{\sigma}$ and $\beta_2 \equiv \frac{1}{1+\eta}\left(\frac{k-(1+\eta)\bar{d}}{\bar{d}}\right)^{\sigma}$.

ceiling is high, so the equilibrium level of one-period securities, s_{12}^*, is zero for a β that is slightly higher than the threshold $1/(1+\eta+\gamma)$. Thus, period-1 agents are unable to reduce s_{12}^* further in response to a decrease in β.

5 Vote on Debt Rule

The analysis, thus far, has assumed that the government takes the fiscal rule, represented by $\bar{d}$, as given. This assumption, which follows Bisin, Lizzeri, and Yariv (2015), is reasonable in the short run, but in the long run there is a tendency toward revising fiscal rules, as described in the Introduction. This section extends the analysis in the previous sections by introducing endogenous determination of the debt rule, and it investigates how the present bias affects, via voting, the design of the debt rules.

For the analysis, we assume that the debt rule is determined before the period-1 selves' decision on saving, s_{12} and s_{13}. Thus, the debt rule is set to maximize the period-1 selves' indirect utility. Within this setting, we consider two cases, $\beta \geq 1/(1+\eta+\gamma)$, which produces no rule breaking, and $\beta < 1/(1+\eta+\gamma)$, which involves the possibility of rule breaking when $\bar{d}$ is given, and we obtain the following result.

Proposition 2 *The optimal debt ceiling for the period-1 selves, $\bar{d}^*$, is*

$$\bar{d}^* \begin{cases} = \bar{d}_L(\beta) & \text{if } 0 \leq \beta < \frac{1}{1+\eta+\gamma}, \\ = \left\{\bar{d}_L(\beta),\ 0\right\} & \text{if } \beta = \frac{1}{1+\eta+\gamma}, \\ = 0 & \text{if } \frac{1}{1+\eta+\gamma} < \beta \leq \frac{1}{1+\eta}, \\ \in [0, k/\eta] & \text{if } \frac{1}{1+\eta} < \beta. \end{cases}$$

Proof. See Appendix A.2.

Figure 6 illustrates the results in Proposition 2, showing that the optimal debt ceiling depends on the degree of present bias, β. When the present bias is weak such that $1/(1+\eta) < \beta$ holds, period-1 agents can curb period-2 agents' excessive consumption, and the corresponding debt issues, only through saving decisions on one- and two-period securities, s_{12} and s_{13}. In other words, debt ceilings are irrelevant for controlling period-2 agents' behavior. Thus, period-1 agents can attain the first-best allocation regardless of the levels of debt ceilings.

When the present bias is strong, such that $\beta \leq 1/(1+\eta)$ holds, period-1 agents cannot curb period-2 agents' behavior only through saving decisions. Debt ceilings are relevant for controlling period-2 agents' excessive consumption and public debt issues. In particular, when β is within the range, $(1/(1+\eta+\gamma), 1/(1+\eta)]$, setting the debt ceiling at $\bar{d}^* = 0$ prevents period-2 agents from choosing excess consumption and public debt.

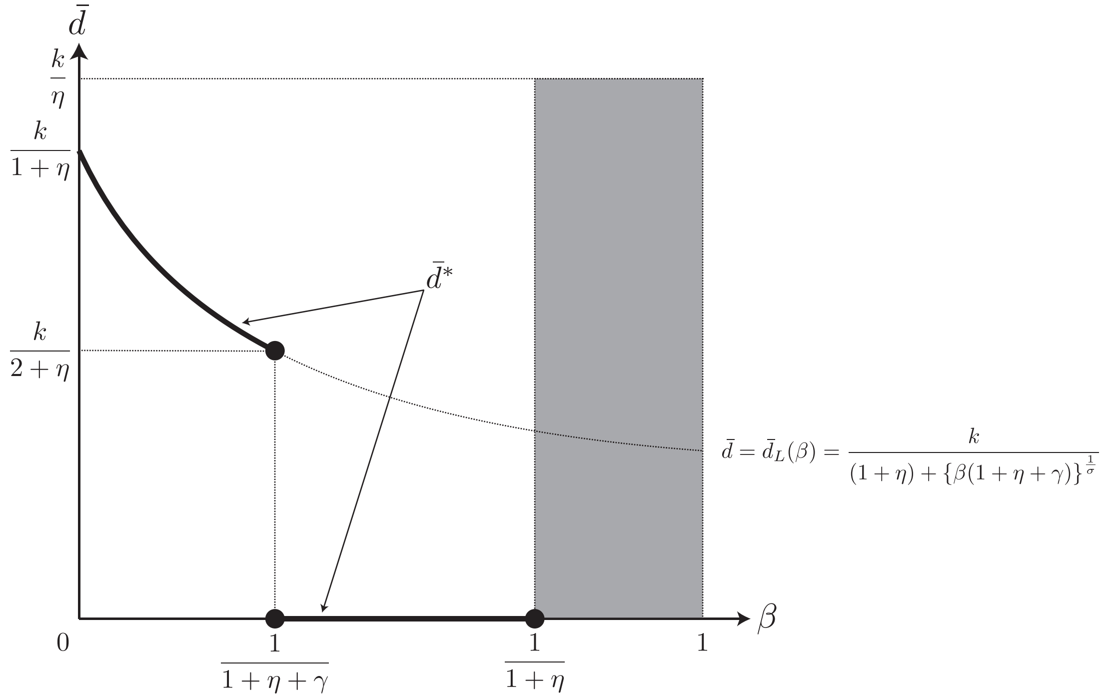

Figure 6: The optimal debt ceiling for the period-1 selves according to β.

Thus, period-1 agents can attain the first-best allocation by managing savings, s_{12} and s_{13}, and debt ceilings, $\bar{d}$.

However, when β is low, such that $\beta < 1/(1+\eta+\gamma)$ holds, setting the debt ceiling at $\bar{d}^* = 0$ induces period-2 agents to break the debt ceiling and thus creates additional costs for period-1 agents. To avoid such cost increases, period-1 agents need to set the debt ceiling at a maximum level such that period-2 agents never break it. This implies that period-2 agents' incentive for rule breaking determines the standard of the rule chosen by period-1 agents. Thus, when the degree of present bias is strong, such that $\beta < 1/(1+\eta+\gamma)$ holds, period-2 agents' excessive consumption is not fully controlled by period-1 agents' decisions on saving and debt ceilings; the first-best allocation is not implementable in this case.

In terms of policy implications, our results suggest that the implementation of debt ceilings contributes to managing possible excessive consumption and the associated overissue of public debt by period-2 agents only for the case with moderate values of $\beta \in (1/(1+\eta+\gamma), 1/(1+\eta)]$. The effectiveness of debt ceilings is limited for the case with low values of β, such that $\beta < 1/(1+\eta+\gamma)$ holds. In this case, the possibility of rule breaking by period-2 agents shapes period-1 agents' decisions on debt ceilings and thus induces them to adopt lax, rather than strict, rules. This implies that, for countries with low values of β, it is inadequate to leave decisions on fiscal rules to each of the countries; rather, it is necessary to induce them to participate in international cooperation characterized by strict standards such as the Maastricht Criteria and the Stability and Growth Pact developed by the European Union.

Another implication of the results is that supranational fiscal rules, like those mentioned above, produce welfare benefits for countries with low values of β, whereas they may create welfare losses for countries with moderate values of β. Countries with low values of $\beta (< 1/(1+\eta+\gamma))$ set lax rules when they each set fiscal rules independently. Lax rules allow the period-2 agents to issue public debt, resulting in a suboptimal allocation. These countries could control overissue of public debt, and thus improve welfare, by participating in international cooperation that imposes further constraints on debt issues. However, countries with moderate values of $\beta \in (1/(1+\eta+\gamma), 1/(1+\eta)]$ set the debt ceiling at $\bar{d}^* = 0$, and thus attain the first-best allocation, when each sets fiscal rules independently. Participating in supranational rules that set debt ceilings at positive levels may induce the period-2 agents in those countries to issue public debt up to the limits, resulting in a suboptimal allocation. Therefore, the choice of target countries is important from the viewpoint of optimality.

6 Conclusion

This paper presented a theoretical framework to examine the political process of determining public debt policy when voters are endowed with present-biased preferences. Specifically, we consider a situation in which debt issue is distortionary and constrained by the debt ceiling, but rule breaking, that is, debt issuance above the ceiling, is available through the incurrence of additional costs. Within this framework, we established that violations of fiscal rules, which are often observed in developed countries, occur when the present bias is strong and the debt ceiling is fairly low. We also studied the endogenous determination of the debt ceiling through voting and showed the debt rule polarization across countries: the debt ceiling is set at zero when the bias is weak, whereas it is set at a positive level when the bias is strong.

The result provides several policy implications for international coordination of fiscal policies, such as that observed within European Union member states. The first result implies that states are more likely to deviate from international coordination, such as the Maastricht criteria, as they become more present biased. The second result implies that international agreements on strict debt rules can be formed and followed only by states endowed with weak present-biased preferences. These implications should be viewed with caution because they are derived using a simple analytical framework. However, they could provide one possible explanation for the success and failure of international agreements on fiscal rules.

A Proofs

A.1 Proof of Proposition 1 and the Equilibrium Allocation

A.1.1 Equilibrium with $d > \bar{d}$

Suppose that the period-1 and -2 selves expect that $d^e = d^c \left(> \bar{d}\right)$ holds. Eq. (8) leads to savings by the period-2 selves, when $d^e = d^c \left(> \bar{d}\right)$ as follows:

$$s^*_{23}(s_{12}, d^c) = \begin{cases} 0 & \text{when } s_{12} \leq S^c(d^c) \\ s^c_{23}(s_{12}, d^c) & \text{when } s_{12} > S^c(d^c) \end{cases} \tag{A.1}$$

where $s^c_{23}(s_{12}, d^c)$ and $S^c(d^c)$ are defined by

$$s^c_{23}(s_{12}, d^c) \equiv s_{12} - \frac{k + \gamma\bar{d} - \left[(1+\eta+\gamma) + (\beta)^{1/\sigma}\right] d^c}{1 + (\beta)^{1/\sigma}},$$

$$S^c(d^c) \equiv \frac{k + \gamma\bar{d} - \left[(1+\eta+\gamma) + (\beta)^{1/\sigma}\right] d^c}{1 + (\beta)^{1/\sigma}}.$$

Figure A.1 illustrates $V_1(s_{12}, d^c)$. When $s_{12} \leq S^c(d^c)$ holds, the first-order condition, with respect to s_{12} in (11), is rewritten as follows:

$$\frac{\partial V_1(s_{12}, d^c)}{\partial s_{12}} = (s_{12} + d^c)^{-\sigma} - \left[k - s_{12} - (1+\eta+\gamma) d^c + \gamma\bar{d}\right]^{-\sigma} \leq 0,$$

where an interior solution is given by

$$s_{12} = \frac{k + \gamma\bar{d} - (2+\eta+\gamma) d^c}{2}.$$

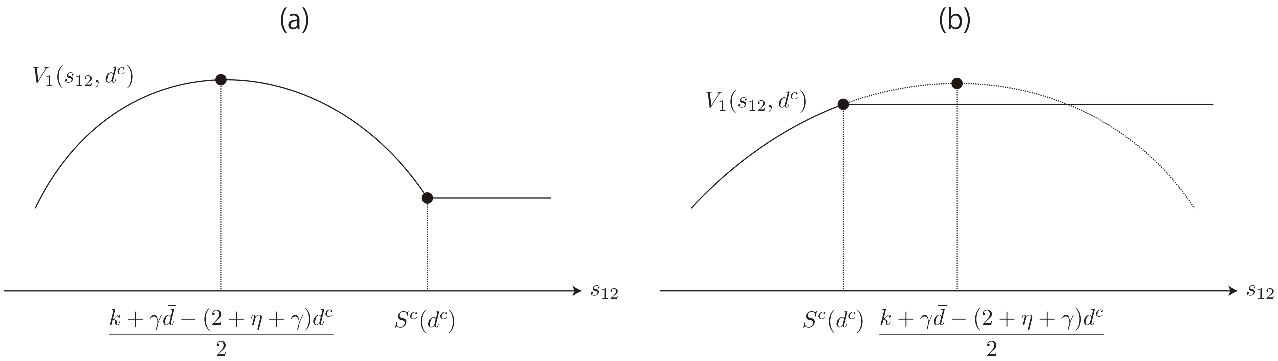

Figure A.1: Illustration of the period-1 selves' utility, $V_1(s_{12}, d^c)$, when $\left[k + \gamma\bar{d} - (2+\eta+\gamma)d^c\right]/2 \leq S^c(d^c)$ (Panel (a)) and $\left[k + \gamma\bar{d} - (2+\eta+\gamma)d^c\right]/2 > S^c(d^c)$ (Panel (b)).

Alternatively, when $s_{12} > S^c(d^c)$ holds, the first-order condition, with respect to s_{12} in (11), becomes

$$\frac{\partial V_1(s_{12}, d^c)}{\partial s_{12}} = 0,$$

suggesting that V_1 is independent of s_{12} as long as $s_{12} > S^c(d^c)$ holds. Notice that $V_1(s_{12}, d^c)$ is continuous at $s_{12} = S^c(d^c)$, as illustrated in Figure A.1.

The interior solution of s_{12} and threshold value $S^c(d^c)$ are compared as follows:

$$S^c(d^c) \gtreqless \frac{k+\gamma\bar{d}-(2+\eta+\gamma)\,d^c}{2} \Leftrightarrow d^c \lesseqgtr \frac{k+\gamma\bar{d}}{\eta+\gamma}.$$

In addition, the following conditions hold:

$$\frac{k+\gamma\bar{d}-(2+\eta+\gamma)\,d^c}{2} \gtreqless 0 \Leftrightarrow d^c \lesseqgtr \frac{k+\gamma\bar{d}}{2+\eta+\gamma},$$
$$S^c(d^c) \gtreqless 0 \Leftrightarrow d^c \lesseqgtr \frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+(\beta)^{1/\sigma}}.$$

Furthermore, the three threshold values of d^c are ranked as

$$\frac{k+\gamma\bar{d}}{2+\eta+\gamma} < \frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+(\beta)^{1/\sigma}} < \frac{k+\gamma\bar{d}}{\eta+\gamma}.$$

The analysis thus far suggests that the allocation of (s_{12}, s_{23}) is given by

$$\begin{array}{rl}
s_{12} > 0, s_{23} = 0 & \text{if } d^c < \frac{k+\gamma\bar{d}}{2+\eta+\gamma} < \frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+(\beta)^{1/\sigma}} < \frac{k+\gamma\bar{d}}{\eta+\gamma}, \\
s_{12} = 0, s_{23} = 0 & \text{if } \frac{k+\gamma\bar{d}}{2+\eta+\gamma} \leq d^c < \frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+(\beta)^{1/\sigma}} < \frac{k+\gamma\bar{d}}{\eta+\gamma}, \\
s_{12} \in [0,k], s_{23} = s_{23}^c(s_{12}, d^c) & \text{if } \frac{k+\gamma\bar{d}}{2+\eta+\gamma} < \frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+(\beta)^{1/\sigma}} \leq d^c < \frac{k+\gamma\bar{d}}{\eta+\gamma}, \\
s_{12} \in [0,k], s_{23} = s_{23}^c(s_{12}, d^c) & \text{if } \frac{k+\gamma\bar{d}}{2+\eta+\gamma} < \frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+(\beta)^{1/\sigma}} < \frac{k+\gamma\bar{d}}{\eta+\gamma} \leq d^c.
\end{array}$$

Based on this classification, the optimal levels of s_{12} and s_{23}, when $d^e = d^c (> \bar{d})$, are given as follows:

$$\begin{array}{lll}
\text{(i)} & s_{12}^* = \frac{k+\gamma\bar{d}-(2+\eta+\gamma)d^c}{2},\ s_{23}^* = 0 & \text{when } d^c < \frac{k+\gamma\bar{d}}{2+\eta+\gamma}, \\
\text{(ii)} & s_{12}^* = 0,\ s_{23}^* = 0 & \text{when } \frac{k+\gamma\bar{d}}{2+\eta+\gamma} \leq d^c < \frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+(\beta)^{1/\sigma}}, \\
\text{(iii)} & s_{12}^* \in [0,k], s_{23}^* = s_{12} - \frac{k+\gamma\bar{d}-\left[(1+\eta+\gamma)+(\beta)^{1/\sigma}\right]d^c}{1+(\beta)^{1/\sigma}} & \text{when } \frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+(\beta)^{1/\sigma}} \leq d^c.
\end{array} \tag{A.2}$$

In what follows, we determine the conditions, such that the expectation of $d^e = d^c (> \bar{d})$ is rational, for the three cases in (A.2).

Case of $d^c < \left(k+\gamma\bar{d}\right)/(2+\eta+\gamma)$

From (4) and (A.2), the expectation of $d^e = d^c$ is rational if the following conditions hold:

$$d^c = \frac{k+\gamma\bar{d}-\left[1+(\beta(1+\eta+\gamma))^{1/\sigma}\right]\frac{k+\gamma\bar{d}-(2+\eta+\gamma)d^c}{2}}{(1+\eta+\gamma)+(\beta(1+\eta+\gamma))^{1/\sigma}} \text{ and } d^c < \frac{k+\gamma\bar{d}}{2+\eta+\gamma},$$

or

$$\left[1-(\beta\,(1+\eta+\gamma))^{1/\sigma}\right]\cdot\left[\left(k+\gamma\bar{d}\right)-(\eta+\gamma)\,d^c\right]=0 \text{ and } d^c<\frac{k+\gamma\bar{d}}{2+\eta+\gamma}. \qquad \text{(A.3)}$$

The first condition in (A.3) indicates that the rational expectation of public debt is given by

$$d^c\begin{cases}\in\left(\bar{d},\frac{k}{\eta}\right) & \text{if } \beta=\frac{1}{1+\eta+\gamma},\\ =\frac{k+\gamma\bar{d}}{\eta+\gamma} & \text{if } \beta\neq\frac{1}{1+\eta+\gamma}.\end{cases}$$

When $\beta\neq 1/(1+\eta+\gamma)$, $d^c=\left(k+\gamma\bar{d}\right)/(\eta+\gamma)$ must satisfy the second condition in (A.3):

$$d^c=\frac{k+\gamma\bar{d}}{\eta+\gamma}<\frac{k+\gamma\bar{d}}{2+\eta+\gamma},$$

but this inequality condition fails to hold. Alternatively, when $\beta=1/(1+\eta+\gamma)$, $d^c\in\left(\bar{d},k/\eta\right)$ with the second condition in (A.3) gives the equilibrium level for rational expectation of public debt as

$$d\in\left(\bar{d},\frac{k+\gamma\bar{d}}{2+\eta+\gamma}\right),$$

where the set is nonempty if $\bar{d}<k/(2+\eta)$.

Proposition A.1 *Suppose that the following conditions hold:*

$$\beta=\frac{1}{1+\eta+\gamma} \textit{ and } \bar{d}<\frac{k}{2+\eta}.$$

There is a rational expectations equilibrium with $d\in\left(\bar{d},\left(k+\gamma\bar{d}\right)/(2+\eta+\gamma)\right)$ *and*

$$\begin{aligned}(c_2,c_3,s_{12},s_{13},s_{23})=&\left(\frac{k+\gamma\bar{d}-(\eta+\gamma)\,d}{2},\frac{k+\gamma\bar{d}-(\eta+\gamma)d}{2},\right.\\ &\left.\frac{k+\gamma\bar{d}-(2+\eta+\gamma)d}{2},\frac{k-\gamma\bar{d}+(2+\eta+\gamma)d}{2},0\right).\end{aligned}$$

Case of $\left(k+\gamma\bar{d}\right)/(2+\eta+\gamma)\leq d^c<\left(k+\gamma\bar{d}\right)/\left(1+\eta+\gamma+(\beta)^{1/\sigma}\right)$

From (4) and (A.2), the expectation of $d^e=d^c$ is rational if the following conditions hold:

$$d^c=\frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+(\beta\,(1+\eta+\gamma))^{1/\sigma}} \text{ and } \frac{k+\gamma\bar{d}}{2+\eta+\gamma}\leq d^c<\frac{k+\gamma\bar{d}}{1+\eta+\gamma+(\beta)^{1/\sigma}}. \qquad \text{(A.4)}$$

This level of public debt is above the limit, $\bar{d}$, if

$$\bar{d}<\frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+(\beta\,(1+\eta+\gamma))^{1/\sigma}}\Leftrightarrow\bar{d}<\frac{k}{(1+\eta)+(\beta\,(1+\eta+\gamma))^{1/\sigma}}=\bar{d}_L(\beta).$$

In addition, d^c must satisfy the second condition in (A.4):

$$\frac{k+\gamma\bar{d}}{2+\eta+\gamma} \leq d^c = \frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+(\beta(1+\eta+\gamma))^{1/\sigma}} < \frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+(\beta)^{1/\sigma}}.$$

The first inequality holds if and only if $\beta \leq 1/(1+\eta+\gamma)$, and the second inequality always holds.

Proposition A.2 *Suppose that the following conditions hold:*

$$\beta \leq \frac{1}{1+\eta+\gamma} \text{ and } \bar{d} < \bar{d}_L(\beta).$$

There is a rational expectations equilibrium with

$$d = \frac{\left(k+\gamma\bar{d}\right)}{(1+\eta+\gamma)+(\beta(1+\eta+\gamma))^{1/\sigma}} \in \left(\bar{d}, \frac{k}{\eta}\right),$$

and

$$(c_2, c_3, s_{12}, s_{13}, s_{23}) = \left(\frac{\left(k+\gamma\bar{d}\right)}{(1+\eta+\gamma)+(\beta(1+\eta+\gamma))^{1/\sigma}}, \frac{(\beta(1+\eta+\gamma))^{1/\sigma}\left(k+\gamma\bar{d}\right)}{(1+\eta+\gamma)+(\beta(1+\eta+\gamma))^{1/\sigma}}, 0, k, 0\right).$$

Case of $\left(k+\gamma\bar{d}\right)/\left(1+\eta+\gamma+(\beta)^{1/\sigma}\right) \leq d^c$

From (4) and (A.2), the expectation of $d^e = d^c$ is rational if the following condition holds:

$$d^c = \frac{\left(k+\gamma\bar{d}\right) - \left[1+(\beta(1+\eta+\gamma))^{1/\sigma}\right]\frac{k+\gamma\bar{d}-\left[(1+\eta+\gamma)+(\beta)^{1/\sigma}\right]d^c}{1+(\beta)^{1/\sigma}}}{(1+\eta+\gamma)+(\beta(1+\eta+\gamma))^{1/\sigma}}$$

and $\left(k+\gamma\bar{d}\right)/\left(1+\eta+\gamma+(\beta)^{1/\sigma}\right) \leq d^c$, or

$$\frac{k+\gamma\bar{d}}{1+\eta+\gamma+(\beta)^{1/\sigma}} \leq d^c = \frac{k+\gamma\bar{d}}{\eta+\gamma}.$$

The associated level of s_{23} is

$$s_{23}^* = s_{12}^* + \frac{k+\gamma\bar{d}}{\eta+\gamma},$$

and the corresponding consumption levels are $c_2 = c_3 = 0$, which contradicts the first-order conditions with respect to c_2 and c_3. Thus, there is no rational expectations equilibrium in this case.

■

A.1.2 Equilibrium with $d = \bar{d}$

Suppose that the period-1 and -2 selves expect that $d^e = \bar{d}$ holds. Equation (8) leads to saving by the period-2 selves when $d^e = \bar{d}$ as follows:

$$s_{23}^*(s_{12}, \bar{d}) = \begin{cases} 0 & \text{when } s_{12} \leq S^u\left(\bar{d}\right), \\ s_{12} - \frac{k - \left[(1+\eta)+(\beta)^{1/\sigma}\right]\cdot \bar{d}}{1+(\beta)^{1/\sigma}} & \text{when } s_{12} > S^u\left(\bar{d}\right), \end{cases}$$

where $S^u\left(\bar{d}\right)$ is defined as

$$S^u\left(\bar{d}\right) \equiv \frac{k - \left[(1+\eta) + (\beta)^{1/\sigma}\right] \cdot \bar{d}}{1 + (\beta)^{1/\sigma}}.$$

Panel (a) of Figure A.2 illustrates $V_1\left(s_{12}, \bar{d}\right)$. When $s_{12} \leq S^u\left(\bar{d}\right)$ holds, the first-order condition, with respect to s_{12} in (11), is rewritten as follows:

$$\frac{\partial V_1\left(s_{12}, \bar{d}\right)}{\partial s_{12}} = \left(s_{12} + \bar{d}\right)^{-\sigma} - \left[k - s_{12} - (1+\eta)\bar{d}\right]^{-\sigma} \leq 0,$$

where an interior solution, given by $s_{12} = \left[k - (2+\eta)\,\bar{d}\right]/2$, is feasible because it satisfies $S^u\left(\bar{d}\right) > \left[k - (2+\eta)\,\bar{d}\right]/2 \Leftrightarrow k/\eta > \bar{d}$. Thus, the optimal level of s_{12} is given by $s_{12} = \left[k - (2+\eta)\,\bar{d}\right]/2$ when $s_{12} \leq S^u\left(\bar{d}\right)$.

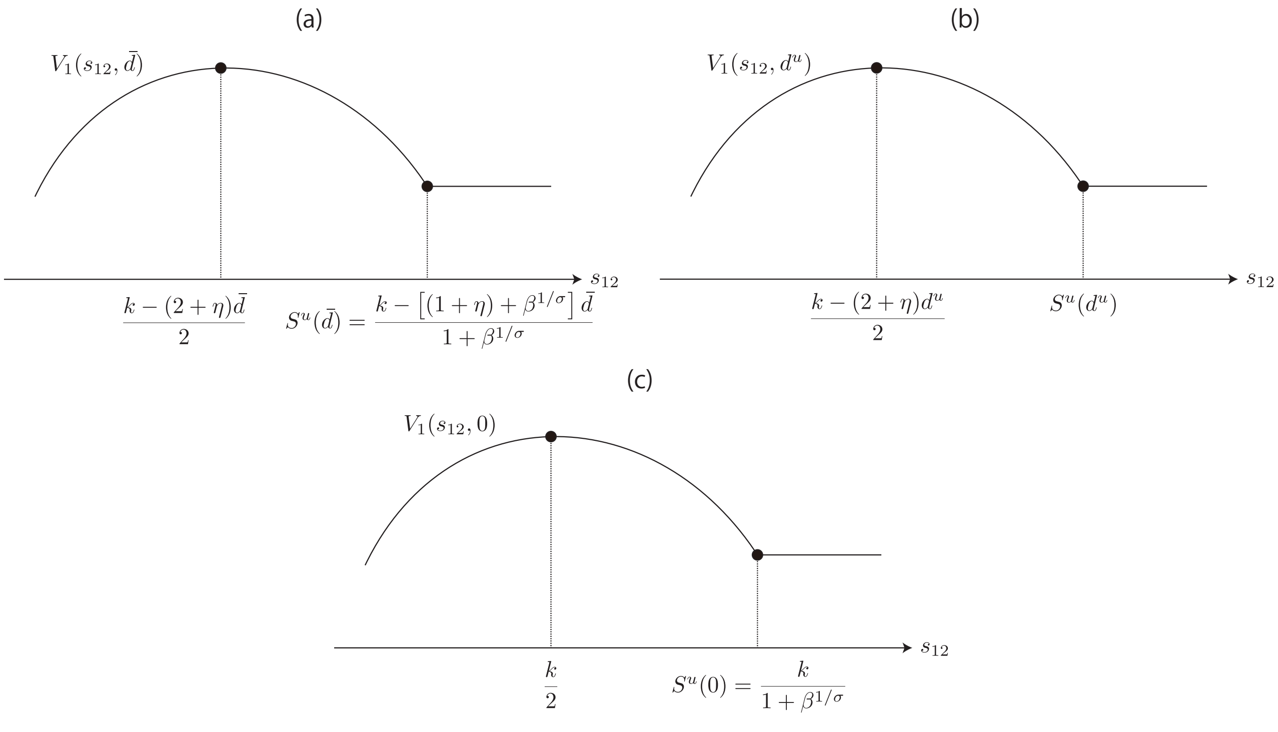

Figure A.2: Illustration of the period-1 selves' utility when $d^e = \bar{d}$ (Panel (a)), $d^e = d^u$ (Panel (b)), and $d^e = 0$ (Panel (c)).

Alternatively, when $s_{12} > S^u\left(\bar{d}\right)$ holds, the first-order condition, with respect to s_{12} in (11), becomes

$$\frac{\partial V_1\left(s_{12}, \bar{d}\right)}{\partial s_{12}} = 0,$$

suggesting that V_1 is independent of s_{12} as long as $s_{12} > S^u\left(\bar{d}\right)$. Notice that V_1 is continuous at $s_{12} = S^u\left(\bar{d}\right)$, and that

$$\frac{k - (2+\eta)\,\bar{d}}{2} > 0 \Leftrightarrow \bar{d} < \frac{k}{2+\eta},$$
$$S^u\left(\bar{d}\right) > 0 \Leftrightarrow \bar{d} < \frac{k}{(1+\eta) + (\beta)^{1/\sigma}}.$$

Given these properties, we can conclude that the optimal levels of s_{12} and s_{23}, when $d^e = \bar{d}$, become:

$$\begin{array}{llll} \text{(i)} & s_{12}^* = \frac{k-(2+\eta)\bar{d}}{2},\, s_{23}^* = 0 & \text{when } \bar{d} < \frac{k}{2+\eta}, & \\ \text{(ii)} & s_{12}^* = 0,\, s_{23}^* = 0 & \text{when } \frac{k}{2+\eta} \le \bar{d} < \frac{k}{(1+\eta)+(\beta)^{1/\sigma}}, & \text{(A.5)} \\ \text{(iii)} & s_{12}^* \in [0,k]\,,\, s_{23}^* = s_{12} - \frac{k-\left[(1+\eta)+(\beta)^{1/\sigma}\right]\bar{d}}{1+(\beta)^{1/\sigma}} & \text{when } \frac{k}{(1+\eta)+(\beta)^{1/\sigma}} \le \bar{d}. & \end{array}$$

In what follows, we determine the conditions, such that the expectation of $d^e = \bar{d}$ is rational, for the three cases in (A.5).

Case of $\bar{d} < k/(2+\eta)$

From (5) and (A.5), the expectation of $d^e = \bar{d}$ is rational if the following conditions hold:

$$A\left(s_{12}^* = \frac{k-(2+\eta)\,\bar{d}}{2}, s_{23}^* = 0\right) \le \bar{d}$$
$$\le d^u\left(s_{12}^* = \frac{k-(2+\eta)\,\bar{d}}{2}, s_{23}^* = 0\right) \text{ and } \bar{d} < \frac{k}{2+\eta}. \qquad \text{(A.6)}$$

The first condition in (A.6) is reformulated as follows:

$$\frac{k - \left[1 + \{\beta\,(1+\eta+\gamma)\}^{1/\sigma}\right]\frac{k-(2+\eta)\bar{d}}{2}}{(1+\eta) + \{\beta\,(1+\eta+\gamma)\}^{1/\sigma}} \le \bar{d} \Leftrightarrow \frac{1}{1+\eta+\gamma} \le \beta,$$

and

$$\bar{d} \le \frac{k - \left[1 + \{\beta\,(1+\eta)\}^{1/\sigma}\right]\frac{k-(2+\eta)\bar{d}}{2}}{(1+\eta) + \{\beta\,(1+\eta)\}^{1/\sigma}} \Leftrightarrow \beta \le \frac{1}{1+\eta}.$$

The equilibrium conditions are summarized in the following proposition.

Proposition A.3 *Suppose that the following conditions hold:*

$$\frac{1}{1+\eta+\gamma} \leq \beta \leq \frac{1}{1+\eta} \text{ and } \bar{d} < \frac{k}{2+\eta}.$$

There is a rational expectations equilibrium with $d = \bar{d}$ and

$$(c_2, c_3, s_{12}, s_{13}, s_{23}) = \left(\frac{k-\eta\bar{d}}{2}, \frac{k-\eta\bar{d}}{2}, \frac{k-(2+\eta)\bar{d}}{2}, \frac{k+(2+\eta)\bar{d}}{2}, 0 \right).$$

Case of $k/(2+\eta) \leq \bar{d} < k/\left[(1+\eta)+(\beta)^{1/\sigma}\right]$

From (5) and (A.5), the expectation of $d^e = \bar{d}$ is rational if the following condition holds:

$$A\left(s^*_{12}=0, s^*_{23}=0\right) \leq \bar{d} \leq d^u\left(s^*_{12}=0, s^*_{23}=0\right) \text{ and}$$
$$\frac{k}{2+\eta} \leq \bar{d} < \frac{k}{(1+\eta)+(\beta)^{1/\sigma}},$$

that is, if

$$\bar{d}_L(\beta) = \frac{k}{(1+\eta)+\{\beta(1+\eta+\gamma)\}^{1/\sigma}} \leq \bar{d} \leq \frac{k}{(1+\eta)+\{\beta(1+\eta)\}^{1/\sigma}} = \bar{d}_H(\beta) \text{ and}$$
$$\frac{k}{2+\eta} \leq \bar{d} < \frac{k}{(1+\eta)+(\beta)^{1/\sigma}}.$$

These are summarized as in the following propositions.

Proposition A.4 *Suppose that the following condition holds:*

$$\max\left\{\frac{k}{2+\eta},\ \bar{d}_L(\beta)\right\} \leq \bar{d} \leq \bar{d}_H(\beta).$$

There is a rational expectations equilibrium with $d = \bar{d}$ and

$$(c_2, c_3, s_{12}, s_{13}, s_{23}) = \left(\bar{d}, k-(1+\eta)\bar{d}, 0, k, 0\right).$$

Case of $k/\left[(1+\eta)+(\beta)^{1/\sigma}\right] \leq \bar{d}$

From (5) and (A.5), the expectation of $d^e = \bar{d}$ is rational if the following condition holds:

$$A\left(s^*_{12}\in[0,k], s^*_{23} = s^*_{12} - \frac{k-\left[(1+\eta)+(\beta)^{1/\sigma}\right]\bar{d}}{1+(\beta)^{1/\sigma}}\right) \leq \bar{d}$$
$$\leq d^u\left(s^*_{12}\in[0,k], s^*_{23} = s^*_{12} - \frac{k-\left[(1+\eta)+(\beta)^{1/\sigma}\right]\bar{d}}{1+(\beta)^{1/\sigma}}\right) \text{ and}$$
$$\frac{k}{(1+\eta)+(\beta)^{1/\sigma}} \leq \bar{d}.$$

The inequality $\bar{d} \leq d^u(\cdot,\cdot)$ is rewritten as

$$\bar{d} \leq \frac{k - \left[1 + \{\beta(1+\eta)\}^{1/\sigma}\right] \frac{k - \left[(1+\eta)+(\beta)^{1/\sigma}\right]\bar{d}}{1+(\beta)^{1/\sigma}}}{(1+\eta) + \{\beta(1+\eta)\}^{1/\sigma}} \Leftrightarrow \eta \leq 0,$$

which fails to hold for any $\eta > 0$. Thus, there is no rational expectations equilibrium with $d = \bar{d}$ when $k/\left[(1+\eta)+(\beta)^{1/\sigma}\right] \leq \bar{d}$.

■

A.1.3 Equilibrium with $d \in (0, \bar{d})$

Suppose that the period-1 and -2 selves expect that $d^e = d^u \in (0, \bar{d})$ holds. Eq. (8) leads to saving by the period-2 selves when $d^e = d^u \in (0, \bar{d})$ as follows:

$$s_{23}^*(s_{12}, d^u) = \begin{cases} 0 & \text{when } s_{12} \leq S^u(d^u), \\ s_{23}^u(s_{12}, d^u) & \text{when } s_{12} > S^u(d^u), \end{cases} \tag{A.7}$$

where $s_{23}^u(s_{12}, d^u)$ and $S^u(d^u)$ are defined as

$$s_{23}^u(s_{12}, d^u) \equiv s_{12} - \frac{k - \left[(1+\eta)+(\beta)^{1/\sigma}\right] d^u}{1+(\beta)^{1/\sigma}},$$

$$S^u(d^u) \equiv \frac{k - \left[(1+\eta)+(\beta)^{1/\sigma}\right] d^u}{1+(\beta)^{1/\sigma}}.$$

Panel (b) of Figure A.2 illustrates $V_1(s_{12}, d^u)$. When $s_{12} \leq S^u(d^u)$ holds, the first-order condition, with respect to s_{12} in (11), is rewritten as follows:

$$\frac{\partial V_1(s_{12}, d^u)}{\partial s_{12}} = (s_{12} + d^u)^{-\sigma} - [k - s_{12} - (1+\eta) d^u]^{-\sigma} \leq 0,$$

where an interior solution, given by $s_{12} = [k - (2+\eta) d^u]/2$, is feasible if it satisfies the following:

$$\frac{k - (2+\eta) d^u}{2} < S^u(d^u) \Leftrightarrow d^u < \frac{k}{\eta}.$$

This condition is satisfied under Assumption 1, $\bar{d} < k/\eta$, and the definition of $d^u (< \bar{d})$.

Alternatively, when $s_{12} > S^u(d^u)$ holds, the first-order condition, with respect to s_{12} in (11), becomes:

$$\frac{\partial V_1(s_{12}, d^u)}{\partial s_{12}} = 0,$$

suggesting that V_1 is independent of s_{12} as long as $s_{12} > S^u(d^u)$ holds. Notice that V_1 is continuous at $s_{12} = S^u(d^u)$ and that

$$\frac{k - (2+\eta) d^u}{2} > 0 \Leftrightarrow d^u < \frac{k}{2+\eta},$$

$$S^u(d^u) > 0 \Leftrightarrow d^u < \frac{k}{(1+\eta)+(\beta)^{1/\sigma}}.$$

Given these properties, we can conclude that the optimal levels of s_{12} and s_{23}, when $d^e = d^u$, become

$$
\begin{array}{llll}
\text{(i)} & s_{12}^* = \frac{k-(2+\eta)d^u}{2}, s_{23}^* = 0 & \text{when } d^u < \frac{k}{2+\eta}, & \\
\text{(ii)} & s_{12}^* = 0, s_{23}^* = 0 & \text{when } \frac{k}{2+\eta} \le d^u < \frac{k}{(1+\eta)+(\beta)^{1/\sigma}}, & \text{(A.8)} \\
\text{(iii)} & s_{12}^* \in [0,k], s_{23}^* = s_{12} - \frac{k-\left[(1+\eta)+(\beta)^{1/\sigma}\right]d^u}{1+(\beta)^{1/\sigma}} & \text{when } \frac{k}{(1+\eta)+(\beta)^{1/\sigma}} \le d^u. &
\end{array}
$$

In what follows, we determine the conditions such that the expectation of $d^e = d^u$ is rational for the three cases in (A.8).

Case of $d^u < k/(2+\eta)$

From (3) and (A.8), the expectation of $d^e = d^u$ is rational if the following conditions hold:

$$
d^u = \frac{k - \left[1 + (\beta(1+\eta))^{1/\sigma}\right] \cdot \left[s_{12}^*(d^u) - s_{23}^*(s_{12}^*(d^u), d^u)\right]}{(1+\eta) + (\beta(1+\eta))^{1/\sigma}} \text{ and } d^u < \frac{k}{2+\eta},
$$

or

$$
\left[1 - (\beta(1+\eta))^{1/\sigma}\right] \cdot \eta d^u = \left[1 - (\beta(1+\eta))^{1/\sigma}\right] \cdot k \text{ and } d^u < \frac{k}{2+\eta}. \qquad \text{(A.9)}
$$

The first condition in (A.9) implies that the rational expectations level of d^u is given by

$$
d^u \begin{cases} \in (0, \bar{d}) & \text{when } \beta(1+\eta) = 1, \\ = k/\eta & \text{when } \beta(1+\eta) \neq 1. \end{cases}
$$

When $\beta(1+\eta) \neq 1$, the candidate for a solution is $d^u = k/\eta$. This candidate is not suitable for the solution because a focus on the case of $d < \bar{d}$ and $\bar{d} < k/\eta$ is assumed in Assumption 1. When $\beta(1+\eta) = 1$, any level of $d^u \in (0, \bar{d})$ with $d^u < k/(2+\eta)$ is rational. Thus, the equilibrium level of public debt becomes

$$
d \in \left(0, \min\left\{\frac{k}{2+\eta}, \bar{d}\right\}\right).
$$

where the set is nonempty if $\bar{d} > 0$.

Proposition A.5 *Suppose that* $\beta(1+\eta) = 1$ *and* $\bar{d} > 0$ *hold. There is a rational expectations equilibrium with* $d \in \left(0, \min\left\{k/(2+\eta), \bar{d}\right\}\right)$ *and*

$$
(c_2, c_3, s_{12}, s_{13}, s_{23}) = \left(\frac{k-\eta d}{2}, \frac{k-\eta d}{2}, \frac{k-(2+\eta)d}{2}, \frac{k+(2+\eta)d}{2}, 0\right).
$$

Case of $k/(2+\eta) \leq d^u < k/\left[(1+\eta)+(\beta)^{1/\sigma}\right]$

From (3) and (A.8), the expectation of $d^e = d^u$ is rational if the following conditions hold:

$$d^u = \frac{k}{(1+\eta)+(\beta(1+\eta))^{1/\sigma}} = \bar{d}_H(\beta) \text{ and } \frac{k}{2+\eta} \leq d^u < \frac{k}{(1+\eta)+(\beta)^{1/\sigma}}$$

or

$$\frac{k}{2+\eta} \leq \bar{d}_H(\beta) < \frac{k}{(1+\eta)+(\beta)^{1/\sigma}}.$$

The first inequality holds if and only if $\beta(1+\eta) < 1$; the second inequality always holds. In addition, d^u must satisfy $d^u < \bar{d}$, that is,

$$d^u = \frac{k}{(1+\eta)+(\beta(1+\eta))^{1/\sigma}} = \bar{d}_H(\beta) < \bar{d}.$$

Proposition A.6 *Suppose that the following conditions hold:*

$$\beta < \frac{1}{1+\eta} \text{ and } \bar{d}_H(\beta) < \bar{d}.$$

There is a rational expectations equilibrium with $d = \bar{d}_H(\beta)$ *and*

$$(c_2, c_3, s_{12}, s_{13}, s_{23}) = \left(\frac{k}{(1+\eta)+(\beta(1+\eta))^{1/\sigma}}, \frac{(\beta(1+\eta))^{1/\sigma}k}{(1+\eta)+(\beta(1+\eta))^{1/\sigma}}, 0, k, 0\right).$$

Case of $k/\left[(1+\eta)+(\beta)^{1/\sigma}\right] \leq d^u$

From (3) and (A.8), the expectation of $d^e = d^u$ is rational if the following conditions hold:

$$d^u = \frac{k - \left[1+(\beta(1+\eta))^{1/\sigma}\right]\cdot[s_{12}^*(d^u) - s_{23}^*(s_{12}^*(d^u), d^u)]}{(1+\eta)+(\beta(1+\eta))^{1/\sigma}} \text{ and } \frac{k}{(1+\eta)+(\beta)^{1/\sigma}} \leq d^u,$$

or

$$d^u = \frac{k - \left[1+(\beta(1+\eta))^{1/\sigma}\right]\cdot\frac{k-\left[(1+\eta)+(\beta)^{1/\sigma}\right]\cdot d^u}{1+(\beta)^{1/\sigma}}}{(1+\eta)+(\beta(1+\eta))^{1/\sigma}} \text{ and } \frac{k}{(1+\eta)+(\beta)^{1/\sigma}} \leq d^u. \quad \text{(A.10)}$$

Solving the first condition in (A.10) for d^u leads to $d^u = k/\eta$. Following the same reasoning as the previous case, this candidate is not suitable for the solution. Thus, there is no rational expectations equilibrium with $k/\left[(1+\eta)+(\beta)^{1/\sigma}\right] \leq d$.

■

A.1.4 Equilibrium with $d = 0$

Suppose that the period-1 and -2 selves expect that $d^e = 0$ holds. Eq. (8) leads to saving by the period-2 selves when $d^e = 0$ as follows:

$$s_{23}^*(s_{12}, 0) = \begin{cases} 0 & \text{when } s_{12} \leq S^u(0) \equiv \frac{k}{1+(\beta)^{1/\sigma}}, \\ s_{23}^u(s_{12}, 0) \equiv s_{12} - \frac{k}{1+(\beta)^{1/\sigma}} & \text{when } s_{12} > S^u(0) \equiv \frac{k}{1+(\beta)^{1/\sigma}}. \end{cases}$$

Panel (c) of Figure A.2 illustrates $V_1(s_{12}, 0)$. When $s_{12} \leq S^u(0)$ holds, the first-order condition, with respect to s_{12} in (11), is rewritten as follows:

$$\frac{\partial V_1(s_{12}, 0)}{\partial s_{12}} = (s_{12})^{-\sigma} - (k - s_{12})^{-\sigma} \leq 0. \tag{A.11}$$

An interior solution, given by $s_{12} = k/2$, is feasible because it holds that $s_{12} = k/2 < S^u(0) \equiv k/\left[1 + (\beta)^{1/\sigma}\right]$. Thus, an optimal level of s_{12} is $s_{12} = k/2$ when $s_{12} \leq S^u(0)$, as illustrated in Panel (a) of Figure A.2.

Alternatively, when $s_{12} > S^u(0)$ holds, the first-order condition, with respect to s_{12} in (11), becomes

$$\frac{\partial V_1(s_{12}, 0)}{\partial s_{12}} = 0,$$

suggesting that V_1 is independent of s_{12} as long as $s_{12} > S^u(0)$ (see Panel (a) of Figure A.2). Notice that V_1 is continuous at $s_{12} = S^u(0)$.

Given the expectation of $d^e = 0$, the optimal level of s_{12} becomes

$$s_{12}^*(0) = \frac{k}{2},$$

and the corresponding level of s_{23} is $s_{23}^*(k/2, 0) = 0$. From (5), the expectation of $d^e = 0$ is rational if the following condition holds:

$$d^u(s_{12}^*, s_{23}^*) = d^u\left(\frac{k}{2}, 0\right) \leq 0 \Leftrightarrow \beta \geq \frac{1}{1+\eta}.$$

Proposition A.7 *Suppose that $\beta \geq 1/(1+\eta)$ holds. There is a rational expectations equilibrium with $d = 0$ and*

$$(c_2, c_3, s_{12}, s_{13}, s_{23}) = \left(\frac{k}{2}, \frac{k}{2}, \frac{k}{2}, \frac{k}{2}, 0\right).$$

■

Up to now, we have shown seven propositions (Propositions A.1 - A.7). In terms of the equilibrium public debt, the results in these propositions are classified in the following way. The equilibrium level of public debt, d^*, satisfies:

(i) $d^* < \bar{d}$ if (a) $\beta = 1/(1+\eta)$ (Proposition A.5), (b) $\beta < 1/(1+\eta)$ and $\bar{d}_H(\beta) < \bar{d}$ (Proposition A.6), or (c) $\beta \geq 1/(1+\eta)$ (Proposition A.7);

(ii) $d^* = \bar{d}$ if (a) $1/(1+\eta+\gamma) \leq \beta \leq 1/(1+\eta)$ and $\bar{d} < k/(2+\eta)$ (Proposition A.3), or (b) $\max\left\{k/(2+\eta), \bar{d}_L(\beta)\right\} \leq \bar{d} \leq \bar{d}_H(\beta)$ (Proposition A.4);

(iii) $d^* > \bar{d}$ if (a) $\beta = 1/(1+\eta+\gamma)$ and $\bar{d} < k/(2+\eta)$ (Proposition A.1), or (b) $\beta \leq 1/(1+\eta+\gamma)$ and $\bar{d} < \bar{d}_L(\beta)$ (Proposition A.2).

We illustrate this classification in Figure 4, taking β in the horizontal axis and $\bar{d}$ in the vertical axis. Focusing on β, we can summarize the results as stated in Proposition 1.

A.2 Proof of Proposition 2

Case of $\beta < 1/(1+\eta+\gamma)$

Suppose that $\beta < 1/(1+\eta+\gamma)$ holds. The equilibrium allocation of consumption for a given $\bar{d}$ is shown in Propositions A.2, A.4, and A.6. It is summarized as follows:

$$(c_2, c_3) = \begin{cases} \left(\frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+\{\beta(1+\eta+\gamma)\}^{1/\sigma}}, \frac{\{\beta(1+\eta+\gamma)\}^{1/\sigma}\left(k+\gamma\bar{d}\right)}{(1+\eta+\gamma)+\{\beta(1+\eta+\gamma)\}^{1/\sigma}}\right) & \text{when } 0 \leq \bar{d} < \bar{d}_L(\beta), \\ \left(\bar{d}, k-(1+\eta)\bar{d}\right) & \text{when } \bar{d}_L(\beta) \leq \bar{d} < \bar{d}_H(\beta), \\ \left(\frac{k}{(1+\eta)+(\beta(1+\eta))^{1/\sigma}}, \frac{(\beta(1+\eta))^{1/\sigma}k}{(1+\eta)+(\beta(1+\eta))^{1/\sigma}}\right) & \text{when } \bar{d}_H(\beta) \leq \bar{d}. \end{cases}$$

The associated indirect utility function of the period-1 selves, $V_1(\bar{d})$, is:

$$V_1(\bar{d}) = \begin{cases} V_{A.2}(\bar{d}) & \text{when } 0 \leq \bar{d} < \bar{d}_L(\beta), \\ V_{A.4}(\bar{d}) & \text{when } \bar{d}_L(\beta) \leq \bar{d} < \bar{d}_H(\beta), \\ V_{A.6} & \text{when } \bar{d}_L(\beta) \leq \bar{d}. \end{cases}$$

where $V_{A.2}(\bar{d})$, $V_{A.4}(\bar{d})$, and $V_{A.6}$ are defined as:

$$V_{A.2}(\bar{d}) \equiv \frac{1}{1-\sigma}\left\{\left[\frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+\{\beta(1+\eta+\gamma)\}^{1/\sigma}}\right]^{1-\sigma}\left[1+\{\beta(1+\eta+\gamma)\}^{(1-\sigma)/\sigma}\right]-2\right\}, \tag{A.12}$$

$$V_{A.4}(\bar{d}) \equiv \frac{1}{1-\sigma}\left\{\left(\bar{d}\right)^{1-\sigma}+\left[k-(1+\eta)\bar{d}\right]^{1-\sigma}-2\right\}, \tag{A.13}$$

$$V_{A.6} \equiv \left\{\left[\frac{k}{(1+\eta)+\{\beta(1+\eta)\}^{1/\sigma}}\right]^{1-\sigma}\left[1+\{\beta(1+\eta)\}^{(1-\sigma)/\sigma}\right]-2\right\}. \tag{A.14}$$

The subscript $j(= A.2, A.4, A.6)$ in the expression of V_j in (A.12), (A.13), and (A.14) implies that the associated allocation of consumption is shown in Proposition j(=A.2, A.4, and A.6), respectively. This notation rule is also applied in the following.

The function $V_1(\bar{d})$ is continuous for $\bar{d} \in (0, \infty)$ because the following properties hold:

$$\lim_{\bar{d}\to\bar{d}_L(\beta)} V_{A.2}(\bar{d}) = V_{A.4}\left(\bar{d}_L(\beta)\right)$$

and

$$\lim_{\bar{d}\to\bar{d}_H(\beta)} V_{A.4}(\bar{d}) = V_{A.6}.$$

In addition, the differentiation of $V_{Ai}(i=2,4,6)$ with respect to $\bar{d}$ leads to

$$\frac{\partial V_{A.2}(\bar{d})}{\partial \bar{d}} = \left[\frac{k+\gamma\bar{d}}{(1+\eta+\gamma)+\{\beta(1+\eta+\gamma)\}^{1/\sigma}}\right]^{-\sigma} \gamma\left[1+\{\beta(1+\eta+\gamma)\}^{(1-\sigma)/\sigma}\right] > 0, \tag{A.15}$$

$$\frac{\partial V_{A.4}(\bar{d})}{\partial \bar{d}} = \left(\bar{d}\right)^{-\sigma} - (1+\eta)\left[k-(1+\eta)\bar{d}\right]^{-\sigma}, \tag{A.16}$$

$$\frac{\partial V_{A.6}}{\partial \bar{d}} = 0, \tag{A.17}$$

where the following condition holds:

$$\frac{\partial V_{A.4}(\bar{d})}{\partial \bar{d}} \gtreqless 0 \Leftrightarrow \bar{d} \lesseqgtr \frac{k}{(1+\eta)+(1+\eta)^{1/\sigma}}. \tag{A.18}$$

Given the assumption of $\beta < 1/(1+\eta+\gamma)$, we have

$$\frac{k}{(1+\eta)+(1+\eta)^{1/\sigma}} < \bar{d}_L(\beta),$$

implying that $V_{A.4}$ is decreasing in $\bar{d}$ for the range of $\left[\bar{d}_L(\beta),\ \bar{d}_H(\beta)\right)$. Thus, the optimal $\bar{d}$ becomes

$$\bar{d}^* = \bar{d}_L(\beta),$$

and the corresponding allocation of saving, consumption, and public debt is given by

$$(s_{12}, s_{13}, c_2, c_3) = \left(0, 0, \frac{k}{(1+\eta)+\{\beta(1+\eta+\gamma)\}^{1/\sigma}}, \frac{\{\beta(1+\eta+\gamma)\}^{1/\sigma} k}{(1+\eta)+\{\beta(1+\eta+\gamma)\}^{1/\sigma}}\right)$$

and $d^* = \bar{d}^*$.

Case of $\beta = 1/(1+\eta+\gamma)$

Based on the results in Propositions A.1, A.2, A.3, A.4, A.6, and the corresponding illustration in Figure 4, we can write the period-1 selves' indirect utility, $V_1(\bar{d})$, as:

$$V_1(\bar{d}) = \begin{cases} \{V_{A.1}(\bar{d}),\ V_{A.2}(\bar{d}),\ V_{A.3}(\bar{d})\} & \text{when } 0 \le \bar{d} < \frac{k}{2+\eta}, \\ V_{A.4}(\bar{d}) & \text{when } \frac{k}{2+\eta} \le \bar{d} \le \frac{k}{(1+\eta)+\left(\frac{1+\eta}{1+\eta+\gamma}\right)^{\frac{1}{\sigma}}}, \\ V_{A.6} & \text{when } \frac{k}{(1+\eta)+\left(\frac{1+\eta}{1+\eta+\gamma}\right)^{\frac{1}{\sigma}}} < \bar{d}, \end{cases}.$$

where $V_{A.2}(\bar{d})$, $V_{A.4}(\bar{d})$, and $V_{A.6}$ are expressed in Eqs. (A.12), (A.13), and (A.14), respectively, and $V_{A.1}(\bar{d})$ and $V_{A.3}(\bar{d})$ are defined as:

$$V_{A.1}(\bar{d}) \equiv \frac{2}{1-\sigma}\left[\left(\frac{k+\gamma\bar{d}-(\eta+\gamma)d}{2}\right)^{1-\sigma}-1\right], \quad d \in \left(\bar{d},\ \frac{k+\gamma\bar{d}}{2+\eta+\gamma}\right), \tag{A.19}$$

$$V_{A.3}(\bar{d}) \equiv \frac{2}{1-\sigma}\left[\left(\frac{k-\eta\bar{d}}{2}\right)^{1-\sigma}-1\right]. \tag{A.20}$$

Conditions in (A.15), (A.16), and (A.17) indicate that for $\bar{d} \geq k/(2+\eta)$, $V_{A.4}(\bar{d})$ is strictly decreasing in $\bar{d}$, and $V_{A.6}$ is independent of $\bar{d}$. Thus, for $\bar{d} \geq k/(2+\eta)$, $V_1(\bar{d})$ is maximized at $\bar{d} = k/(2+\eta)$, and the corresponding value of $V_1(\bar{d})$ is

$$\max_{\bar{d}\geq k/(2+\eta)} V_1(\bar{d}) = V_{A,4}\left(\frac{k}{2+\eta}\right) = \frac{2}{1-\sigma}\left[\left(\frac{k}{2+\eta}\right)^{1-\sigma}-1\right].$$

Next, consider the range of $\bar{d}$ such that $0 \leq \bar{d} < k/(2+\eta)$. For this range, there are multiple equilibria presented in Propositions A.1, A.2, and A.3, depending on the expectations of the period-1 selves. In what follows, we specify the maximized value of $V_1(\bar{d})$ for each expectation.

(i) Case of Proposition A.1

Suppose that for $\bar{d} \in [0, k/(2+\eta)]$, the period-1 selves expect the allocation presented in Proposition A.1. There are continuum levels of the equilibrium public debt, d, such that $\bar{d} < d < (k+\gamma\bar{d})/(2+\eta+\gamma)$. This implies that the indirect utility, $V_{A.1}(\bar{d})$, is dependent on the realization of d. In what follows, we show that $V_{A.1}(\bar{d}) < V_{A.4}(k/(2+\eta))$ holds for any $d \in \left(\bar{d}, (k+\gamma\bar{d})/(2+\eta+\gamma)\right)$, that is, $V_1(\bar{d})$ is maximized at $\bar{d} = k/(2+\eta)$.

Assume that the agents in periods 1 and 2 have the same expectation for d and that their expectation is unaffected by the choice of $\bar{d}$ in period 1. Under this assumption, $V_{A.1}(\bar{d})$ in Eq. (A.19) is strictly increasing in $\bar{d}$. Then, we have

$$\begin{aligned}\sup_{0\leq\bar{d}<k/(2+\eta)} V_{A,1}(\bar{d}) &= \lim_{\bar{d}\to\frac{k}{2+\eta}} V_{A,1}(\bar{d}) \\ &= \frac{2}{1-\sigma}\cdot\left[\left(\frac{k+\frac{\gamma k}{2+\eta}-(\eta+\gamma)d}{2}\right)^{1-\sigma}-1\right],\end{aligned}$$

where the expression in the second line is strictly decreasing in d.

Notice that $\inf d \to k/(2+\eta)$ when $\bar{d} \to k/(2+\eta)$. Given this property, we obtain

$$\begin{aligned}
\sup_{0 \le \bar{d} < k/(2+\eta)} V_{A.1}(\bar{d}) &< \frac{2}{1-\sigma} \cdot \left[\left(\frac{k + \frac{\gamma k}{2+\eta} - (\eta+\gamma)\frac{k}{2+\eta}}{2} \right)^{1-\sigma} - 1 \right] \\
&= \frac{2}{1-\sigma} \cdot \left[\left(\frac{k}{2+\eta} \right)^{1-\sigma} - 1 \right] \\
&= V_{A.4} \left(\frac{k}{2+\eta} \right).
\end{aligned}$$

This indicates that $V_{A.1}(\bar{d}) < V_{A.4}\left(k/(2+\eta)\right)$ holds for any $\bar{d} \in [0, k/(2+\eta)]$. Therefore, we conclude that the optimal $\bar{d}$ is $\bar{d}^* = k/(2+\eta)$ when the period-1 selves expect the allocation presented in Proposition A.1.

(ii) Case of Proposition A.2

Suppose that for $\bar{d} \in [0, k/(2+\eta))$, the period-1 selves expect the allocation presented in Proposition A.2. The indirect utility function of the period-1 selves is $V_{A.2}(\bar{d})$ in Eq. (A.12). Given the assumption of $\beta = 1/(1+\eta+\gamma)$, the expression in Eq. (A.12) is reduced to:

$$V_{A,2}(\bar{d}) = \frac{2}{1-\sigma} \cdot \left[\left(\frac{k + \gamma \bar{d}}{2+\eta+\gamma} \right)^{1-\sigma} - 1 \right].$$

Since $V_{A,2}(\bar{d})$ is strictly increasing in $\bar{d}$, we have

$$\begin{aligned}
\sup_{0 \le \bar{d} < k/(2+\eta)} V_{A,2}(\bar{d}) &= \lim_{\bar{d} \to \frac{k}{2+\eta}} V_{A,2}(\bar{d}) \\
&= \frac{2}{1-\sigma} \cdot \left[\left(\frac{k}{2+\eta} \right)^{1-\sigma} - 1 \right] \\
&= V_{A,4} \left(\frac{k}{2+\eta} \right).
\end{aligned}$$

This implies that $V_{A,2}(\bar{d}) < V_{A,4}\left(k/(2+\eta)\right)$ holds for any $\bar{d} \in [0, k/(2+\eta))$. Therefore, we conclude that the optimal $\bar{d}$ is $\bar{d}^* = k/(2+\eta)$ when the period-1 selves expect the allocation presented in Proposition A.2.

(iii) Case of Proposition A.3

Suppose that for $\bar{d} \in [0, k/(2+\eta))$, the period-1 selves expect the allocation presented in Proposition A.3. The indirect utility function of the period-1 selves is $V_{A.3}(\bar{d})$ in Eq.

(A.20). The function $V_{A.3}(\bar{d})$ is strictly decreasing in $\bar{d}$. For $\bar{d} \in [0, k/(2+\eta))$, we have

$$\begin{aligned}
\max_{0 \leq \bar{d} < k/(2+\eta)} V_{A,3}(\bar{d}) &= V_{A,3}(0) \\
&= \frac{2}{1-\sigma} \cdot \left[\left(\frac{k}{2} \right)^{1-\sigma} - 1 \right] \\
&> \frac{2}{1-\sigma} \cdot \left[\left(\frac{k}{2+\eta} \right)^{1-\sigma} - 1 \right] \\
&= V_{A,4} \left(\frac{k}{2+\eta} \right).
\end{aligned}$$

Therefore, we conclude that the optimal $\bar{d}$ is $\bar{d}^* = 0$ when the period-1 selves expect the allocation presented in Proposition A.3.

Case of $1/(1+\eta+\gamma) < \beta \leq 1/(1+\eta)$

Suppose that the ceiling is set at $\bar{d} = 0$. The corresponding allocation of consumption, from Proposition A.3, is

$$(c_2, c_3) = \left(\frac{k}{2}, \frac{k}{2} \right).$$

This allocation of consumption is consistent with the solution to the following period-1 selves' utility maximization problem:

$$\max \frac{(c_2)^{1-\sigma}}{1-\sigma} + \frac{(c_3)^{1-\sigma}}{1-\sigma} \text{ s.t. } c_2 + c_3 \leq k.$$

Thus, the optimal level of $\bar{d}$ becomes $\bar{d}^* = 0$.

Case of $\beta > 1/(1+\eta)$

When $\beta > 1/(1+\eta)$, the equilibrium allocation of consumption, from Proposition A.7, is

$$(c_2, c_3) = \left(\frac{k}{2}, \frac{k}{2} \right) \quad \forall \, \bar{d} \in \left[0, \frac{k}{\eta} \right).$$

Since the allocation is independent from $\bar{d}$, the optimal level of $\bar{d}$ becomes $\bar{d}^* \in [0, k/\eta)$.

■

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