The Political Economy of Entitlement Protection under Sovereign Default Risk*

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September 7, 2026

Working Paper Series No.68
Faculty of Economics, Doshisha University

Abstract

Sovereign default often extends beyond the repudiation of government debt to cuts in entitlements such as public pensions. How far entitlements are protected in default—the degree of entitlement protection—varies widely across countries, reflecting the legal status of benefit rights, yet the consequences of this variation have remained unexplored. This paper builds an overlapping-generations model of a two-period economy that features both politically determined entitlements and endogenous sovereign default, and conducts comparative statics with respect to the degree of protection. The degree of protection governs the composition of the two instruments through which the politically empowered young extract resources from the future. Weaker protection inflates entitlement promises, whose burden shrinks in default, whereas stronger protection inflates debt issuance. As a result, the equilibrium default probability is non-monotonic in the degree of protection: both too little and too much protection raise default risk. Moreover, a marginal adjustment of protection in the direction that reduces the default probability raises the welfare of both the generation that issues the debt and the generation that inherits it. The design of entitlement protection is thus not a zero-sum intergenerational redistribution but a policy instrument that can improve fiscal stability and the welfare of both generations at once.

Key words: Government debt; Entitlements; Default; Political economy.

JEL Classification: D72, E62, H60, H63

*I am grateful to Shinya Fujita, Makoto Hanazono, Toshiki Tamai, Wataru Tamura, Takayuki Tsuruga, and Akihiko Yanase for helpful comments, and I thank the seminar participants at Nagoya University. Any remaining errors are my own. Financial support from the Japan Society for the Promotion of Science Grants-in-Aid for Scientific Research (C) (No. 25K05011) is gratefully acknowledged. The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper. In preparing this manuscript, the author used a generative AI tool (Claude, Anthropic) to assist with editing, refining the exposition, and checking the consistency of the text. All analytical results, proofs, and conclusions are the author's own. The author reviewed and verified all AI-assisted material and takes full responsibility for the content of this article.

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1 Introduction

Concerns about fiscal sustainability have been mounting in recent years, particularly in emerging market and middle-income economies. Indeed, as of 2024, roughly 51% of emerging market economies were running primary deficits in excess of the level required to stabilize their debt (IMF 2025). According to estimates by the International Monetary Fund (IMF), if current policies continue, the gross debt-to-GDP ratio of emerging market and middle-income economies is projected to rise from about 49% in 2016 to about 88% by 2030 (Figure 1). Public debt continues to mount worldwide as well: it is projected to reach about 94% of GDP in 2025 and 100% in 2029 (IMF 2026).

Among the pressures behind this debt accumulation, the growth of entitlement-related spending has attracted particular attention in recent years. The IMF defines an entitlement as "any spending program where expenditure is open-ended (usually transfer/grant payments) and where recipients must be paid or given transfers/grants if they meet certain criteria" (IMF 2025); public pensions, public health insurance, and unemployment insurance are leading examples. Because of their sheer size and the difficulty of cutting them, entitlements are often called "implicit government debt" and are a key determinant of fiscal sustainability. Smetters (2004) and Kotlikoff (2006) show, both theoretically and empirically, that the institutional rigidity of entitlement programs undermines the flexibility of fiscal policy and amplifies long-run fiscal risk.

Figure 2 shows, for the 31 emerging market economies for which data are available, the share of government expenditure devoted to the sum of health and social protection spending—the core components of entitlement spending—as of 2019.[1] The simple average across the 31 countries is 35.4%, and in some countries—Poland, Brazil, and Moldova among them—roughly half of government expenditure is devoted to entitlements.

What matters here is that this large scale and the difficulty of cutting entitlements noted above are two sides of the same coin. That a large share of government expenditure is institutionally locked in as benefits owed to people who have already established eligibility means that, for future governments, the resources available for discretionary use are reduced in advance by exactly that amount. The structure

[1] The Government Expenditures by Function classification in the IMF's Government Finance Statistics contains no category called "entitlements." This paper uses the sum of Health and Social protection as a proxy for entitlement spending. Social protection includes public pensions as well as unemployment, disability, family, and housing benefits, and thus corresponds closely to the definition of entitlements used in this paper. Health, on the other hand, includes not only public health insurance benefits but also discretionary spending such as hospital construction. Note, therefore, that this proxy captures a somewhat broader set of programs than entitlements in the strict sense. All figures used below are on a general government basis, covering social security funds in addition to central, state, and local governments. This coverage is necessary because most pension benefits are paid out of social security funds. The year 2019 is chosen because the temporary expansion of benefits during the COVID-19 pandemic substantially distorts the figures from 2020 onward.

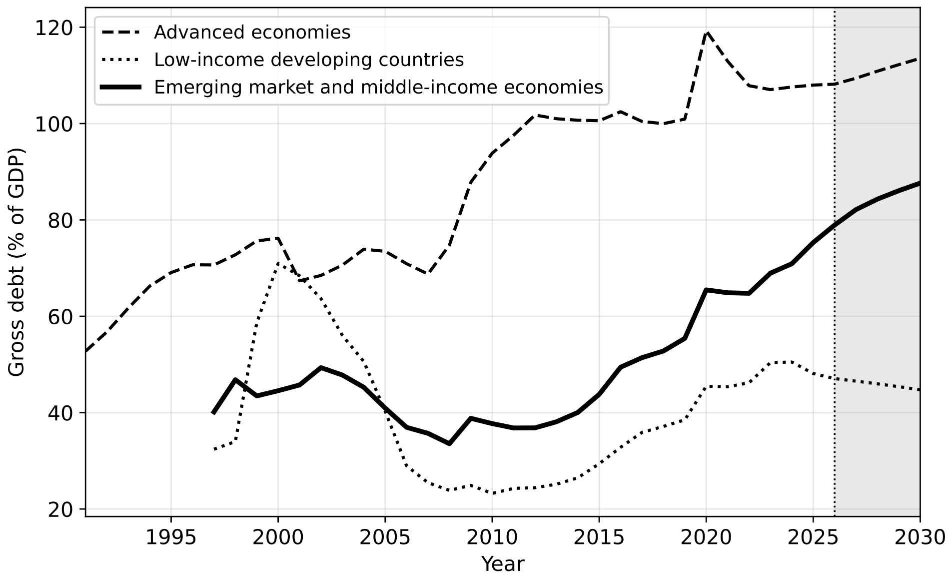

Figure 1: Gross debt position by income group (% of GDP, 1991–2030). The thick solid line denotes emerging market and middle-income economies, the dashed line advanced economies, and the dotted line low-income developing countries. The shaded area indicates the projection period (2027 onward, IMF estimates). Debt is in principle on a general government basis, but for some emerging market and middle-income economies and low-income developing countries it is on a central government (budgetary central government) basis, following the IMF classification. Data source: IMF, Fiscal Monitor (April 2026).

is the same as that of debt service, which pre-commits future resources. In this sense, entitlements are first-order fiscal obligations on a par with government debt, and assessing fiscal sustainability requires understanding how the size of entitlements is determined and how entitlements and debt issuance shape each other.

Against this background, the political economy literature has made substantial progress in analyzing fiscal outcomes while distinguishing between discretionary and entitlement spending. Baron (1996) and Bowen et al. (2014) use dynamic bargaining models to show that entitlements can generate a persistent expansion of public spending through their status quo effect. Azzimonti et al. (2023) show that rising income inequality strengthens the bargaining power of the poor and contributes to the long-run growth of entitlement spending. Bouton et al. (2020) point out that entitlements can be strategic complements to government debt, and Piguillem and Riboni (2024) show that the stickiness of entitlement spending generates a non-linear relationship with debt accumulation.[2]

These studies have deepened our understanding of how entitlements shape fiscal obligations. Most of them, however, implicitly assume that government debt and entitlements, once set, will surely be honored in the future, leaving outside their scope the possibility of default that is frequently observed

[2]Section 2 reviews the related literature in more detail.

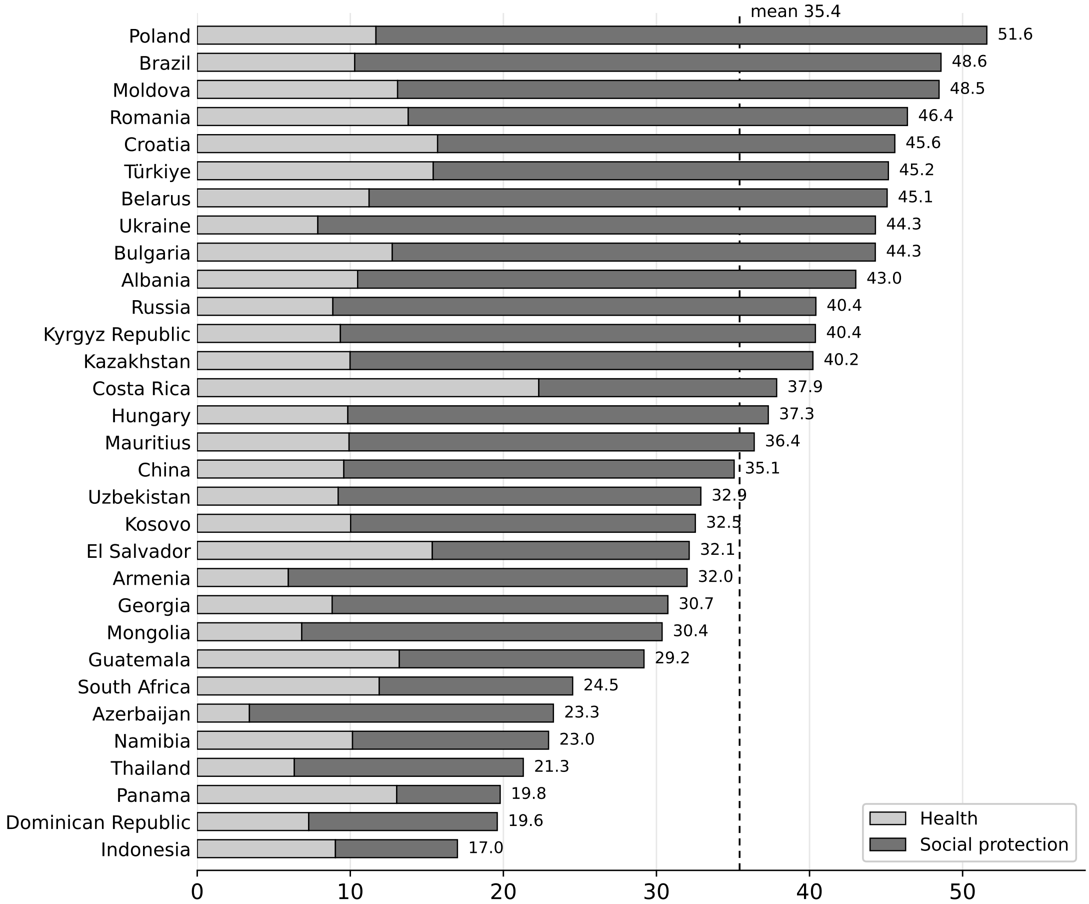

Figure 2: A proxy for entitlement spending in emerging market economies (% of total government expenditure, 2019). The IMF statistics contain no category called "entitlements"; the sum of Health and Social protection is used here as a proxy. Light gray denotes Health and dark gray Social protection; the number to the right of each bar is their sum (%). All expenditures are on a general government basis, covering social security funds in addition to central, state, and local governments. Data source: IMF, Government Finance Statistics (Government Expenditures by Function).

in emerging market economies. In reality, sovereign defaults are not rare. According to Erce et al. (2024), between 1980 and 2018 there were 116 defaults on local-currency debt and 177 on foreign-currency debt, of which 87 and 118, respectively, were by emerging market economies. Their analysis estimates the annual frequency of domestic debt defaults at 2.1% and that of external debt defaults at 3.0%—sovereign default risk is far from negligible.

Moreover, when default or severe fiscal distress occurs, repayment obligations on government debt are not the only claims that are cut. Entitlements promised in advance, such as pensions, are not necessarily honored either, and often become the object of cuts or institutional overhaul. In Greece, a partial default on government bonds took place in 2012, and the fiscal adjustment surrounding it included cuts in pension benefits, notably reductions in accrual rates (Clements et al. 2013). Russia fell into severe fiscal distress throughout the 1990s, defaulting on its government debt in 1998. Under that distress the government was unable to pay promised public pensions for extended periods, and at one point about 14 million of the roughly 39 million pensioners received no benefits at all (Jensen and Richter 2003). As these episodes show, entitlements are not necessarily protected under default or deep fiscal crisis, and the extent of the cuts varies: they may be confined to part of the benefits, as in Greece, or extend to nearly all of them in the form of prolonged non-payment, as in Russia.

Taking these two facts into account—that sovereign defaults do occur, and that entitlements are also cut when they do—is essential for a political economy analysis of debt and entitlements concerned with fiscal sustainability. Bouton et al. (2020), on which this paper builds, themselves single out the incorporation of sovereign default as an important avenue for future research in their concluding remarks. Both debt issuance and entitlement promises raise the default probability by squeezing future budgets; at the same time, the very possibility of default feeds back into how both are determined.

The possibility of default, however, affects the two heterogeneous fiscal obligations—government debt and entitlements—asymmetrically. Debt can be sold only at a price discounted for anticipated default, which restrains borrowing. Entitlements, by contrast, are not traded in markets, so no such price-based discipline operates on them. What determines the burden that promising entitlements places on future budgets is how much of them is cut in default. If the cut is deep, then even inflated promises need be honored only in part when default occurs, and the check on setting large entitlements is correspondingly weak. In short, debt and entitlements respond to the possibility of default in different ways. The level of debt, the level of entitlements, and the default risk they generate are thus mutually determined. An analysis that leaves sovereign default and the accompanying entitlement cuts out of its scope misses this interdependence from the outset.

The purpose of this paper is to introduce sovereign default, and the entitlement cuts that accompany it, into the political economy analysis of entitlements. These two elements have so far been treated in separate literatures. Political economy models with explicit entitlements presuppose a world in which promised benefits are always honored, while the sovereign default literature has been concerned exclusively with debt repayment and has not analyzed entitlements. To the best of my knowledge, no existing model features both politically determined entitlements and endogenous sovereign default, with entitlements subject to cuts upon default.

Because the two have been studied separately, the question of how far entitlements are protected in default has never been posed. This is precisely the question on which this paper focuses. As the episodes above illustrate, the extent to which entitlements are cut under default varies widely. This paper explicitly incorporates into the model the fraction of promised entitlements that is guaranteed in default, the degree of entitlement protection $m \in [0, 1]$.[3] The case $m = 1$ corresponds to entitlements being fully guaranteed even in default, and a smaller m represents weaker protection. Existing political economy models that presuppose a world without default implicitly set $m = 1$.

What pins down m in reality is, first, the legal status with which benefit rights are designed—whether they are protected by the constitution, protected as property rights, or treated as mere gratuities (Monahan 2010)—and, second, how far the judiciary actually upholds that protection in times of fiscal crisis. As a result, m differs greatly across countries and states. Neither determinant is something a government can adjust nimbly in response to current fiscal conditions, but neither is an immutable feature of the environment: through legal reform, m can be redesigned in the long run. This is why this paper conducts comparative statics with respect to m. Section 8 examines these institutions and the concrete differences in m they generate—across U.S. states and across European countries during the euro crisis.

This paper takes as its foundation the overlapping-generations model of a two-period economy in Bouton et al. (2020), in which both debt issuance and entitlement setting are endogenous, introduces endogenous sovereign default following Arellano (2008), and conducts comparative statics with respect to $m \in [0, 1]$. The model is structured as follows. Consider a two-period economy with two groups, young and old, in each period. The group holding political power in period 1 (the issuing generation) issues government debt to borrow resources from the future and, at the same time, sets entitlements, thereby reserving in advance the transfer it will receive in period 2, when it is old. Debt and entitlements thus both serve as instruments through which the issuing generation extracts resources from the future holder of political power. The government provides public goods in each period, and the issuing generation also derives benefits from period-2 public goods. Period-2 public goods are financed by what remains of that period's income after debt repayment and entitlement payments.

The group holding political power in period 2 (the successor generation) takes the inherited levels of debt and entitlements as given and chooses whether or not to default on the debt. Choosing default

[3] Entitlement cuts are not limited to those accompanying sovereign default. Like government debt, entitlements have also been cut ex ante through reforms of pension systems and related programs. Since the 1990s, for example, the Czech Republic, Lithuania, and Georgia have raised the eligibility age for public pensions (Holzmann and Hinz 2005), and Bulgaria has implemented similar reforms since the early 2000s (IMF 2012). In Argentina, the pension indexation formula was changed in 2017 from one linked to wages and tax revenues to one combining consumer price inflation and wage growth (Carrera and Angelaki 2021). These are reforms designed to contain entitlement spending and improve the sustainability of pension systems, and their effect can be interpreted as a partial reneging on pre-promised entitlements. In advanced economies, where population aging and rising longevity are further along, such entitlement cuts are observed even more frequently. In Japan, the eligibility age for public pensions has been raised in stages to 65 (Clements et al. 2014), and other advanced economies have undertaken large-scale pension reforms, including increases in the eligibility age (Italy and Spain), reduced incentives for early retirement (Denmark and Italy), and cuts in pension benefits (Greece and Portugal) (IMF 2012). The focus of this paper is not on such preemptive cuts but on the ex post cuts that accompany sovereign default.

wipes out the debt but lowers income, and only the fraction $m \in [0,1]$ of the entitlements set by the issuing generation remains guaranteed; the remaining fraction $(1-m)$ is not paid out and is instead retained in the period-2 government's budget. Because period-2 income is a random variable, default occurs stochastically; the default probability is endogenously determined by the levels of debt and entitlements and by the degree of protection m, and is reflected in the price of government bonds.

Within this framework, the paper analyzes two relationships: between the degree of entitlement protection m and the sovereign default probability, and between m and the welfare of each generation. The main results are as follows. The first is that the relationship between the degree of entitlement protection m and the equilibrium default probability is non-monotonic. On the one hand, near the lower bound at which protection is so weak that debt issuance falls to zero (which I call the borrowing shutdown level), strengthening protection (raising m) lowers the equilibrium default probability. On the other hand, near full protection ($m = 1$), relaxing protection (lowering m) lowers the default probability. That is, both too little and too much protection raise the default probability, so the degree of protection that minimizes default risk lies at neither endpoint but strictly between them.

This non-monotonicity arises because the degree of protection shifts the composition of the instruments through which the issuing generation extracts resources from the future. What determines the period-2 government's incentive to default is the total fiscal outlay saved by defaulting. Debt is wiped out in full, whereas only the unprotected portion of entitlements is written off. When protection is weak, most of the promised entitlements are cut in default. Put differently, adding one unit to the promise obliges the government to actually pay only the fraction m in default, so the addition weighs little on period-2 resources. Extraction through entitlements is cheap in this sense, and the issuing generation sets them high. At the same time, because default under weak protection deeply cuts the entitlements the issuing generation itself receives, borrowing that raises the default probability does not pay, and debt issuance stays small.

When protection is strong, this relationship is exactly reversed: entitlements become expensive and are restrained, while the issuing generation, whose own entitlements are protected even in default, willingly accepts a higher default probability and increases debt issuance. Thus protection that is too weak inflates entitlements and protection that is too strong inflates debt, and at either extreme the fiscal outlay saved by default becomes large.

The second main result is that adjustments of protection in the direction that lowers the default probability can be Pareto improving, sacrificing neither generation. Specifically, near each of the two endpoints above, a marginal change in m that lowers the default probability—strengthening protection at the lower end, relaxing it at the upper end—raises the welfare of the issuing generation and the successor generation simultaneously.

Why both generations gain at once is easiest to see in the case of relaxing protection near full protection. The successor generation bears the consequences of default, so a relaxation of protection that lowers the default probability raises its welfare. What is counterintuitive is that even the issuing generation—the party that set the entitlements—gains from the relaxation. Near full protection, however, its own entitlements are already almost fully protected, so little benefit remains from strengthening protection further. In default, on the other hand, nearly all of the scarce resources are absorbed by guaranteed entitlement payments and little is left to finance public goods, so at the margin public goods are more valuable. Relaxing protection and redirecting resources toward public goods is therefore preferable for the issuing generation itself. Near the lower end, a symmetric logic implies that strengthening protection raises the welfare of both generations.

These two results deliver implications that point in the same direction for policy decisions on entitlement protection. First, from the standpoint of minimizing default risk, protection should be neither too strong nor too weak. Second, adjustments in that direction—strengthening protection at the lower end, relaxing it at the upper end—come at the expense of neither generation, because the fall in the default probability raises the welfare of both the current and the future generation. In intergenerational political economy, the choice of the degree of entitlement protection tends to be framed as a zero-sum redistribution problem; the results of this paper show that this framing is not necessarily correct.

Moreover, the real-world policy levers that move m—how far benefit rights are protected by constitutions and statutes, and how judicial review in times of crisis is designed—correspond directly to the parameter targeted by the comparative statics of this paper. In this sense, the paper offers a new perspective on the design of fiscal institutions: the legal protection of pensions and social security can serve as a policy instrument that improves fiscal stability and intergenerational welfare at the same time.

The remainder of the paper is organized as follows. Section 2 reviews the related literature and clarifies the paper's position within it. Section 3 formalizes the political and economic environment, including sovereign default and the associated possibility of entitlement cuts, and presents the model. Section 4 defines the equilibrium concept and derives the equilibrium. Section 5 analyzes the effect of the degree of entitlement protection on the equilibrium default probability. Section 6 builds on this to analyze its effect on the welfare of each generation. Section 7 uses numerical examples to examine the role of the parameter conditions underlying these two results and delineates the parameter range over which they hold. Section 8 discusses the institutional determinants of m in practice and their international diversity. Section 9 concludes.

2 Related literature

2.1 The political economy of government debt and intergenerational conflict

This paper is related to the political economy literature on government debt and intergenerational conflict. This literature has shown that the group holding political power may use government debt strategically, both to shift resources across time and to constrain the choices of future policymakers with different preferences. Alesina and Tabellini (1990) show that when policymakers with conflicting preferences alternate in office, the incumbent has an incentive to issue debt strategically in order to influence the fiscal policy of its successor. Similarly, Persson and Svensson (1989) show that a conservative government may strategically choose fiscal deficits to constrain the spending of a future government with different preferences. Tabellini and Alesina (1990) extend this insight to a majority-voting environment, showing that the current majority chooses deficits to tilt future fiscal decisions in its favor.

Whereas these studies emphasize the strategic over-accumulation of government debt, Song et al. (2012) point to a political mechanism that disciplines debt accumulation. In their overlapping-generations model, successive generations vote on public goods, taxes, and debt. Debt allows the current generation to shift the fiscal burden onto the future, but the young oppose large debt because it crowds out the provision of public goods in their own old age. This opposition restrains debt accumulation in equilibrium—a restraining force they call the "disciplining effect" of young voters.

This paper belongs directly to this tradition. As in these models, the young generation holding political power in period 1 (the issuing generation) uses government debt to extract resources from the future. Like the young voters in Song et al. (2012), moreover, this generation internalizes the fact that issuing more debt reduces future public goods provision, which restrains its borrowing. The paper departs from the existing literature in two respects. First, it introduces entitlements as a second instrument of intergenerational redistribution alongside debt, allowing the politically empowered generation to extract resources from the future through two distinct channels. Second, it allows the future government to default endogenously on the debt and, in a setting where default occurs stochastically, analyzes how the degree of entitlement protection affects the default probability and intergenerational welfare.

2.2 The political economy of entitlements

This paper also contributes to the political economy literature on entitlements. Baron (1996) and Bowen et al. (2014) focus on public programs that persist unless new legislation is passed—entitlements, regulations, and the like—and use dynamic political economy models to analyze the mechanisms through which such programs are chosen. In these studies, policy in each period is determined through political bargaining based on take-it-or-leave-it budget proposals. As a result, the level of public goods provision implemented in the previous period serves as an endogenous status quo in the next period's bargaining. This structure gives the governing party an incentive to set public spending above its statically optimal level and can generate a persistent, cumulative expansion of public spending.

Azzimonti et al. (2023) aim to explain the long-run expansion of entitlement spending in the United States. They construct a dynamic political economy model in which two parties, representing the interests of the rich and the poor respectively, bargain over taxes, entitlement spending, and public goods provision. Their theoretical and quantitative analysis shows that "unbalanced growth" characterized by rising income inequality strengthens the bargaining power of the poor through the status quo mechanism, causing entitlement spending to grow over time.

Other studies focus on the interaction between entitlement spending and government debt. A leading early contribution analyzing their strategic complementarity in a political economy framework is Bouton et al. (2020). They show that under capital market frictions, tightening fiscal rules that limit government borrowing can strategically increase the government's reliance on entitlement spending. As a result, "total government obligations," defined as the sum of government debt and entitlement spending, expand, potentially generating outcomes that are Pareto inferior for all parties.

Piguillem and Riboni (2024), in turn, analyze the effect of the stickiness of entitlement spending on debt accumulation. In their model, the incumbent government is required to maintain a fixed fraction of the previous period's spending level, a constraint formalized as "spending stickiness." Their analysis shows that the relationship between spending stickiness and debt accumulation is hump-shaped, so that the effect of stickiness on debt dynamics differs greatly with its degree.

In sum, the existing political economy literature has been mainly concerned with uncovering the mechanisms through which entitlement spending becomes excessive and with analyzing its interaction with government debt. Most of these studies, however, implicitly assume that government debt is always repaid and entitlements are never cut, and do not analyze how the possibility of sovereign default—in particular, how far entitlements are protected in default—affects default risk or intergenerational welfare. Building on this literature, this paper takes a step forward by explicitly introducing the degree of entitlement protection m and analyzing how its magnitude affects the equilibrium default probability and the welfare of each generation. Whereas existing studies, including Bouton et al. (2020), assume that entitlements are honored in full ($m = 1$), this paper extends the analysis to the case $m < 1$.

2.3 The political economy of sovereign default

This paper is also related to the literature on the political economy of sovereign default. Theoretical research on sovereign default has developed a variety of models to clarify the mechanisms that generate sovereign risk and their macroeconomic consequences. Among the earliest contributions in this line is Eaton and Gersovitz (1981), whose framework provides the theoretical foundation for credit constraints in sovereign debt markets and serves as the starting point for subsequent quantitative analyses of sovereign default.

Since the 2000s, a prominent strand of the sovereign default literature has used DSGE models to quantitatively analyze the relationship between sovereign default and business cycles in emerging market economies. Aguiar and Gopinath (2006), for example, introduce regime shocks to the trend growth rate of output and show that their framework can account for the countercyclical interest rates and default frequencies observed in emerging market economies. Arellano (2008) constructs a DSGE model with endogenous default risk and shows, in line with the empirical evidence, that high debt levels in recessions strengthen the incentive to default. Mendoza and Yue (2012) present a general equilibrium model that jointly analyzes sovereign default and business cycle dynamics, identifying a financial mechanism that operates at default and amplifies fluctuations.

A substantial body of work also focuses on the lack of commitment inherent in international lending relationships. Cole and Kehoe (2000) formalize the concept of self-fulfilling debt crises in a general equilibrium framework, showing that crises can involve multiple equilibrium regions—one in which no crisis occurs, one in which a crisis may occur, and one in which only default occurs. Their analysis is pioneering in showing that crises can arise not only from fundamentals but also from self-fulfilling shifts in market expectations. Dovis (2019) shows that in a production economy with informational and commitment frictions, default and temporary exclusion from international financial markets arise endogenously along the equilibrium path as the outcome of efficient risk sharing.

Within the sovereign default literature, some studies focus on partial default, that is, "haircuts."

Adam and Grill (2017), for example, show that even when the government can fully commit, partially deviating from its legal repayment obligations—choosing partial default—can be optimal under certain conditions. Amador and Phelan (2021, 2023) introduce reputation as a central determinant of default decisions, constructing a new theoretical framework that incorporates government types and market belief formation. Arellano et al. (2023) identify a mechanism through which partial default today induces default in the future.

More recently, studies have emerged that explicitly link sovereign default to the political process. Scholl (2024) introduces a two-party system and probabilistic voting into a quantitative sovereign debt model with heterogeneous households and non-linear income taxation, analyzing how distributional and electoral motives over redistribution shape default incentives. That paper shows that electoral uncertainty can raise equilibrium default risk, and shares with this paper the aim of endogenizing the policy decisions behind default as a political process.

In short, research on sovereign default has developed along three lines: quantitative analysis based on DSGE models, theoretical work emphasizing commitment problems and the size and form of default, and political economy analysis that explicitly incorporates the political process. This paper relates to this literature by adopting the endogenous default probability framework in the spirit of Arellano (2008), in which period-2 income is a random variable and the period-2 government decides ex post whether or not to default. That is, unlike analyses of partial default in which the size of the default (the haircut rate) is chosen endogenously, this paper relies on a full-default framework in which the probability of default is determined endogenously.

The contribution of this paper is to embed this framework in a political economy model, introduce entitlements alongside government debt, and analyze how the degree of entitlement protection m affects the equilibrium default probability and intergenerational welfare. In doing so, the paper adds a new political economy perspective—entitlement protection—to the existing research on the determinants of sovereign risk.

3 Model

This section presents the model. The model builds on the overlapping-generations model of a two-period economy in Bouton et al. (2020), with three modifications. First, period-2 income is a random variable. Second, the period-2 government is given the option to default on the debt. Third, households in period 1 cannot save or borrow privately at all; that is, they face an extreme form of capital market imperfection. The following subsections describe, in turn, the demographic structure and the economy, the sovereign default setup, the budget constraints and preferences of households and the government, and the resource constraints that consolidate them.

3.1 Demographics and the economy

Demographics

Consider a two-period model with three groups, which we name according to their roles as follows. The initial old are the old generation that exists only in period 1. The issuing generation is the generation that, when young in period 1, issues government debt d and sets entitlements y^E, and then bears the consequences as the old in period 2. The successor generation inherits the debt d and entitlements y^E left by the issuing generation and, as the period-2 government, decides whether to repay the debt or default. Thus in period 1 the issuing generation works while the initial old are retired, and in period 2 the successor generation works while the issuing generation is retired. Of the three groups, only the issuing generation makes intertemporal decisions. Figure 3 summarizes this demographic structure and the roles in each period.

Political decisions are assumed to be made by each period's workers, that is, the young generation. The model focuses on intergenerational political frictions and abstracts from intragenerational ones. The assumption that workers make political decisions is a natural one: workers typically outnumber pensioners, so the median voter by age is a worker. Indeed, Galasso and Profeta (2004) and Galasso (2008) estimate the age of the median voter using cross-country data on demographics and political participation, and show that even after accounting for the relatively high turnout of the elderly, the median voter in France, Italy, the United Kingdom, and the United States around 2000 was in their forties. The assumption is thus empirically plausible as well.

Consumption, saving, and policy variables

Workers supply labor inelastically. The issuing generation's income when young is deterministic and equal to $z_1 \geq 0$. The successor generation's income when young, z_2, is a random variable drawn from the uniform distribution $U[\underline{z}, \overline{z}]$ $(0 \leq \underline{z} < \overline{z})$.

The period-1 government levies a lump-sum tax τ_1 on workers and issues government debt d. The initial stock of debt at the beginning of period 1 is assumed to be zero. Debt is a one-period bond with price q: the government issues d units of bonds in period 1, receives qd units of goods, and repays d units of goods in period 2. Bonds are traded in the international capital market, and $d < 0$—government saving—is allowed, although the analysis below focuses on equilibria with $d \geq 0$.

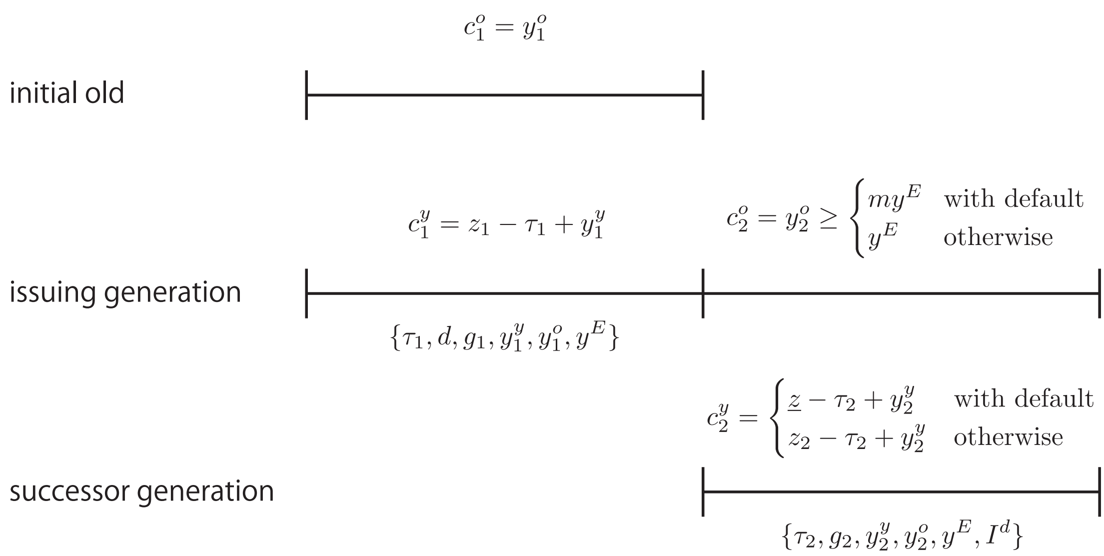

Figure 3: The structure of generations.

The period-1 government spends the resources thus obtained on transfers y_1^i ($i = y, o$ denoting the young and the old, respectively), which may differ across generations, and on public goods g_1. In addition, the period-1 government can set in advance a floor on the transfer that the issuing generation will receive as the old in period 2, $y_2^o \geq y^E$. This paper calls this pre-promised minimum transfer an "entitlement." The period-2 government cannot freely renege on this promise; as described below, a cut occurs only if it chooses to default.

The period-2 government levies a lump-sum tax τ_2 on workers and repays the debt d issued in period 1, although, as described below, it can instead choose to default. Since period 2 is the final period of the economy, no new debt is issued. The period-2 government also spends on transfers y_2^i and public goods g_2.

Households face capital market imperfections and are assumed to be unable to save or borrow privately at all, as in Arellano (2008). This setup corresponds to the limiting case of Bouton et al. (2020), in which households can save and borrow only on terms less favorable than those on government bonds—the limit in which that disadvantage is large enough that households cease private saving and borrowing altogether. Under this assumption, households receive tax cuts and income transfers through debt issuance in place of private borrowing, and secure period-2 income by setting entitlements in place of private saving.

3.2 Sovereign default

The consequences of default

In period 1 the government trades its bonds in the international financial market. Its counterparties in the market are assumed to have the following three properties: (i) they are risk neutral and competitive; (ii) they know the true distribution of households' period-2 income z_2; and (iii) they can buy and sell a safe bond with gross interest rate R at any time.

When the government issues debt, lenders cannot enforce its repayment; that is, the government can default. Default is assumed to be full default (Arellano 2008; Azzimonti and Mitra 2023): choosing default wipes out the debt entirely, and creditors recover nothing.

When the government chooses to default, two things are assumed to happen. First, period-2 income falls from z_2 to the lower bound of the income distribution, $\underline{z}$. This can be interpreted in terms of the setup in Arellano (2008), where a defaulting government is excluded from international financial markets for some time and output falls during the exclusion. The drop of period-2 income to $\underline{z}$ in this paper can be read as the output loss associated with such market exclusion.

Second, entitlements are cut from y^E to my^E, where $m \in [0, 1]$ is the parameter representing the fraction of entitlements that remains guaranteed in default; a larger m means more strongly protected entitlements. The size of the cut is $(1 - m)y^E$, and the amount cut is not used to repay the debt but is absorbed into the period-2 government's general budget (public goods provision, spending on the young, and so on). This can be viewed as the fiscal adjustment the government needs in order to sustain its spending when income falls sharply in default.

The degree of entitlement protection m, which governs the size of the cut, is an institutional parameter capturing how far entitlements are protected under fiscal stress. In reality, m is determined by the legal status with which benefit rights are designed—whether they are protected by the constitution, protected as property rights, or treated as mere gratuities (Monahan 2010)—and by how far the judiciary actually upholds that protection in times of fiscal crisis; its level differs greatly across countries and states (see Section 8 for details). The reason this paper treats m as an exogenous institutional parameter and only d and y^E as endogenous choices is that the two operate on very different time scales. Whereas debt issuance and entitlement setting are chosen by each period's government through budgetary and legislative processes, the constitutions and judicial doctrines that determine m change far more slowly and only at great cost.

Entitlements themselves also have an institutional character, but they sit at a different tier from

m. While y^E is the level of the promise chosen by the government at a point in time, m is the rule that determines how far such promises are honored under fiscal stress, and belongs to the legal system rather than to any individual program. Nor is y^E a freely adjustable variable within the model: once set, the period-2 government can reduce it only by the fraction $(1 - m)$, and only through default.

These two consequences of default—the fall in income and the cut in entitlements—mean different things to the period-2 government that chooses whether to default, whose preferences are those of the successor generation. The fall in income is a cost borne by the successor generation itself: by destroying $z_2 - \underline{z}$ units of resources, it deters default. The entitlement cut, by contrast, is not a cost to them. Since the amount cut, $(1 - m)y^E$, flows into the general budget, it is a benefit of default on a par with the extinction of the repayment obligation d, and it is larger the smaller the degree of protection m. The party that bears the entitlement cut as a cost is the issuing generation, which is old in period 2.

Default risk and the interest rate

Let δ denote the probability that the government defaults on its debt. Because the possibility of default affects the bond price (the interest rate on government debt), the model must incorporate the interest rate explicitly. Under the full-default assumption, creditors recover nothing when default occurs. Thus when the government issues d units of bonds, lenders invest qd in period 1 and, in period 2, recover d with probability $1 - \delta$ and receive nothing with probability δ. For risk-neutral lenders, this expected return must equal the return qdR from investing the same funds in the safe bond. The bond price q is therefore determined by the arbitrage condition

$$q = \frac{1 - \delta}{R}. \tag{1}$$

If default risk is zero, that is, $\delta = 0$, then $q = 1/R$. Default risk δ is an endogenous variable that depends on d, y^E, and the degree of entitlement protection m; its explicit form is derived below.

3.3 Households

Let c_t^i denote the consumption of generation i in period t. In period 1, the budget constraints of the initial old and the issuing generation are given by

$$c_1^o = y_1^o,$$

$$c_1^y = z_1 - \tau_1 + y_1^y.$$

Since private saving and borrowing are unavailable, all disposable income when young is devoted to current consumption.

In period 2, the budget constraints of the issuing generation (old) and the successor generation (young) depend on whether default occurs and are given by

$$c_2^o = y_2^o \geq \begin{cases} my^E & \text{with default} \\ y^E & \text{otherwise} \end{cases}$$

$$c_2^y = \begin{cases} \underline{z} - \tau_2 + y_2^y & \text{with default} \\ z_2 - \tau_2 + y_2^y & \text{otherwise} \end{cases}$$

where $z_2 \sim U\left[\underline{z}, \overline{z}\right]$. The issuing generation's old-age consumption is financed solely by the transfer y_2^o it receives when old. The period-2 government cannot make a transfer below the entitlement set in advance by the period-1 government. If no default occurs, this lower bound is the pre-set level y^E itself; if default occurs, the fraction $(1-m)$ of the entitlement is cut, so the lower bound falls to my^E.

The successor generation's consumption when young, c_2^y, reflects the fact that income is $\underline{z}$ in default and z_2 otherwise. Which generation consumes in which period corresponds to the structure in Figure 3.

Log utility is assumed throughout. The period utility function is given by

$$u\left(c_t^i, g_t\right) = \ln c_t^i + \alpha \ln g_t,$$

where $\alpha > 0$ is the parameter representing the relative importance of public goods.

Each generation's lifetime utility is the sum of these period utilities over the periods it lives. The issuing generation's lifetime utility is $u(c_1^y, g_1) + \beta E\left[u(c_2^o, g_2)\right]$, where $\beta \in (0,1)$ is the discount factor. The expectation is taken over the realization of period-2 income z_2; whether default occurs is also decided by the period-2 government in response to that realization. The initial old's utility is $u(c_1^o, g_1)$ and the successor generation's is $u(c_2^y, g_2)$. Since the economy ends after two periods, the successor generation's utility consists only of its utility when young.

3.4 Government

The government's budget constraints are as follows. The period-1 government's budget constraint is

$$g_1 + y_1^y + y_1^o = qd + \tau_1.$$

The period-2 government's budget constraint depends on whether default occurs and is given by

$$\begin{cases} g_2 + y_2^o + y_2^y = \tau_2 \text{ \& } y_2^o \geq my^E & \text{with default} \\ g_2 + y_2^o + y_2^y + d = \tau_2 \text{ \& } y_2^o \geq y^E & \text{otherwise} \end{cases}$$

Here the last constraint, $y_2^o \geq my^E$ or $y_2^o \geq y^E$, represents the entitlement: the period-1 government can set in advance a floor on the income transfer that the issuing generation, old in period 2, will receive. If the period-2 government defaults, the entitlement is cut by $(1-m)y^E$, and the amount cut is not used to repay the debt but is absorbed into the period-2 government's general budget (public goods provision, support for the young, and so on).[4] Because default is full, repayment to creditors is zero.

Period-2 policy is chosen by the successor generation, for whom transfers to the old are resources that would otherwise have gone to their own consumption or to public goods. They therefore have no incentive to transfer more than the entitlement floor, and the constraint always binds with equality. The two equations above can then be rewritten as

$$\begin{cases} g_2 + my^E + y_2^y = \tau_2 & \text{with default} \\ g_2 + y^E + y_2^y + d = \tau_2 & \text{otherwise} \end{cases}$$

3.5 Consolidating the problems and the resource constraints

Taxation in this model is non-distortionary and there is no heterogeneity within groups. The government's problem in each period can therefore be merged with that of the period's median voter—the issuing generation in period 1 and the successor generation in period 2. Combining the household and government budget constraints in each period yields the resource constraints of the consolidated

[4] An extreme real-world example of this "cut flowing into the general budget" aspect is Argentina's pension nationalization. Shut out of international financial markets after its 2001 default, and seeking fiscal resources, the country in 2008 seized the accumulated assets of the private pension funds (AFJPs) and absorbed them into the public pension system (Bouton et al. 2020; Carrera and Angelaki 2021; Clements et al. 2013). Although this was an expropriation of assets rather than a cut in benefit levels per se, it corresponds to the inflow of $(1-m)y^E$ into the general budget in this model, in that a fiscally distressed government diverted resources earmarked for entitlements to its general budget.

problem. The period-1 resource constraint is

$$g_1 + c_1^y + c_1^o = z_1 + qd, \tag{2}$$

and the period-2 resource constraint, depending on whether default occurs, is

$$\begin{cases} g_2 + my^E + c_2^y = \underline{z} & \text{with default} \\ g_2 + y^E + c_2^y = z_2 - d & \text{otherwise} \end{cases} \tag{3}$$

4 Equilibrium analysis

This section first defines the equilibrium concept used in this paper, then lays out the timing of events in the model, and derives the equilibrium by backward induction.

4.1 Equilibrium definition and timing

The equilibrium concept used in this paper is defined as follows.

Definition 1. *An equilibrium of this model is a set of allocations, a bond price, and policies determined in period 1,*

$$\left\{d^*,\ y^{E*},\ q^*,\ \tau_1^*,\ g_1^*,\ c_1^{y*},\ c_1^{o*},\ y_1^{y*},\ y_1^{o*}\right\}$$

together with allocations and policies that depend on the realization of period-2 income z_2,

$$\left\{I^{d*}(z_2),\ \tau_2^*(z_2),\ g_2^*(z_2),\ c_2^{y*}(z_2),\ c_2^{o*}(z_2),\ y_2^{y*}(z_2),\ y_2^{o*}(z_2)\right\}$$

satisfying the following conditions, where $I^{d*}(z_2)$ is the indicator function that equals 1 if the period-2 government chooses to default and 0 otherwise.

(i) *Taking the period-2 responses $\{I^{d*}, g_2^*\}$ and the bond price $q(d, y^E)$ as given, the period-1 government and households choose $\{d^*, y^{E*}, \tau_1^*, g_1^*, c_1^{y*}, c_1^{o*}, y_1^{y*}, y_1^{o*}\}$ to maximize the issuing generation's lifetime utility subject to the period-1 resource constraint (2).*

(ii) *For each realization of period-2 income z_2, taking $\{d^*, y^{E*}\}$ as given, the period-2 government and households choose the default decision I^{d*} and the allocations and policies $\{\tau_2^*, g_2^*, c_2^{y*}, c_2^{o*}, y_2^{y*}, y_2^{o*}\}$*

to maximize the successor generation's lifetime utility subject to the period-2 resource constraint (3)*.*

(iii) *The bond price satisfies the arbitrage condition* (1)*, that is,* $q^* = (1-\delta^*)/R$*, where* δ^* *is the equilibrium default probability implied by the default decision in (ii).*

(iv) *All household budget constraints, the government budget constraints, and the resource constraints* (2) *and* (3) *hold.*

Next, the timing of events in the model is as follows. Because the decisions of the government and households in each period affect subsequent choices and equilibrium conditions, the problem must be solved backward. The sequence of events is:

1. The period-1 government chooses $\{d, y^E, \tau_1, g_1, y_1^y, y_1^o\}$, internalizing the period-2 responses $\{I^d, g_2\}$ and the bond price schedule $q(d, y^E)$ determined by the arbitrage condition (1). The equilibrium bond price is simultaneously determined as $q^* = q(d^*, y^{E*})$.
2. The issuing generation's consumption c_1^y and the initial old's consumption c_1^o are determined.
3. z_2 is observed.
4. Taking $\{d, y^E\}$ as given, the period-2 government decides whether or not to default.
5. The period-2 government chooses $\{\tau_2, g_2, y_2^y, y_2^o\}$.
6. The successor generation's consumption c_2^y and the issuing generation's old-age consumption c_2^o are determined.

The problem is solved backward following this sequence.

The following assumption is maintained throughout the analysis.

Assumption 1. $R = \beta = 1$.

The same assumption is made in Bouton et al. (2020). It is a natural specification with no discounting, and allowing the interest rate to exceed the discount rate, or introducing discounting, would not change the results qualitatively. Under this assumption, fully smoothing consumption across periods is optimal; hence, if full smoothing fails to obtain, the cause lies in political frictions.

4.2 The period-2 government and household problem

4.2.1 The period-2 problem under repayment

First, consider the successor generation's problem in period 2 when the government does not default (the consolidated problem of the private sector and the government). It is written as

$$\begin{aligned} \max_{c_2^y, g_2} \quad & \ln c_2^y + \alpha \ln g_2 \\ \text{s.t.} \quad & g_2 + y^E + c_2^y = z_2 - d, \\ & d, y^E, z_2 \text{ given.} \end{aligned} \tag{4}$$

Solving this problem yields the policy functions

$$c_{2,r}^{y*}(z_2, d, y^E) = (1 - A)X_r(z_2, d, y^E), \quad g_{2,r}^*(z_2, d, y^E) = AX_r(z_2, d, y^E), \tag{5}$$

where the subscript r denotes the repayment case and

$$\begin{aligned} X_r(z_2, d, y^E) &\equiv z_2 - d - y^E, \\ A &\equiv \frac{\alpha}{1+\alpha}. \end{aligned} \tag{6}$$

The term $X_r(z_2, d, y^E)$ is what remains of period-2 income z_2 after subtracting the repayment and benefit obligations, $d + y^E$, that the issuing generation leaves to the period-2 government, and represents the resources the period-2 government can use freely if it does not default—hereafter called *fiscal space*—while A is the share of that fiscal space allocated to public goods.

An expansion of $\{d, y^E\}$ raises these obligations and shrinks fiscal space X_r by the same amount. Hence, as (5) shows, both the successor generation's consumption and period-2 public goods provision are decreasing in $\{d, y^E\}$: the larger the repayment and benefit obligations reserved in advance by the issuing generation, the more the successor generation must cut back its own consumption and public goods. The analysis below proceeds under the assumption that $\{d, y^E\}$ lie in the range where $c_{2,r}^{y*} > 0$ and $g_{2,r}^* > 0$; that this condition holds in equilibrium is verified later.

Finally, substituting (5) into the objective function (4) yields the period-2 value under repayment. Throughout, V denotes a generation's lifetime (indirect) utility, with the subscript indicating the generation; S stands for the successor generation, which exists only in period 2, so its lifetime utility coincides with its utility when young.

$$V_S^r(z_2, d, y^E) = (1+\alpha)\ln X_r(z_2, d, y^E) + \ln(1-A) + \alpha \ln A. \tag{7}$$

The last two terms on the right-hand side are constants independent of z_2, d, y^E.

4.2.2 The period-2 problem under default

Next, consider the successor generation's problem when the government defaults in period 2. Full default wipes out the debt entirely, while period-2 income falls from z_2 to the lower bound of the distribution $\underline{z}$ and entitlements are reduced from y^E to my^E. The problem is written as

$$\begin{aligned} \max_{c_2^y, g_2} \quad & \ln c_2^y + \alpha \ln g_2 \\ \text{s.t.} \quad & g_2 + my^E + c_2^y = \underline{z}, \\ & y^E \text{ given.} \end{aligned} \tag{8}$$

Solving this problem yields the policy functions

$$c_{2,d}^{y*} = (1-A)X_d(y^E), \quad g_{2,d}^* = AX_d(y^E), \tag{9}$$

where the subscript d denotes the default case and

$$X_d(y^E) \equiv \underline{z} - my^E. \tag{10}$$

Here $X_d(y^E)$ represents period-2 fiscal space when default is chosen. Since the debt is wiped out entirely while income is fixed at $\underline{z}$, X_d depends on neither z_2 nor d. The analysis below proceeds under the assumption that y^E lies in the range where $c_{2,d}^{y*} > 0$ and $g_{2,d}^* > 0$; that this condition holds in equilibrium is verified later.

Substituting (9) into the objective function (8) yields the period-2 value under default, which is a function of y^E alone.

$$V_S^d(y^E) = (1+\alpha)\ln X_d(y^E) + \ln(1-A) + \alpha \ln A. \tag{11}$$

4.2.3 The default decision and the default probability

The period-2 government chooses to default whenever $V_S^d(y^E) > V_S^r(z_2, d, y^E)$. From (7) and (11), this condition can be rewritten as

$$V_S^d(y^E) > V_S^r(z_2, d, y^E) \;\Leftrightarrow\; X_d(y^E) > X_r(z_2, d, y^E) \;\Leftrightarrow\; \underline{z} + d + (1-m)y^E > z_2.$$

Default is chosen when fiscal space under default, X_d, exceeds fiscal space under repayment, X_r. Since X_r is smaller the lower the realization of period-2 income z_2 and the larger the inherited obligations $d + y^E$, default is chosen when income turns out low or when the issuing generation's extraction is large. The two instruments of extraction, however, do not operate with equal force. Because default wipes out the debt in full while cutting entitlements only by $(1-m)y^E$, a unit of y^E widens the range of incomes over which default is chosen by only $1-m$ times as much as a unit of d. Under full protection, $m = 1$, expanding entitlements leaves the default incentive unchanged.

The default decision is therefore represented by the indicator function

$$I^{d*} = \begin{cases} 1 & \text{if } \hat{z}(d, y^E) > z_2 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

where the default threshold is defined as

$$\hat{z}(d, y^E) \equiv \underline{z} + d + (1-m)y^E. \tag{13}$$

Finally, consider the probability, as seen from period 1, that the period-2 government defaults on the debt. Using (12), (13), and $z_2 \sim U\left[\underline{z}, \overline{z}\right]$, the default probability δ is given by

$$\delta(d, y^E) = \begin{cases} 1 & \text{if } \hat{z}(d, y^E) \geq \overline{z} \\ p_z\left[d + (1-m)y^E\right] & \text{if } \hat{z}(d, y^E) \in (\underline{z}, \overline{z}) \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

where p_z is the probability density of period-2 income z_2, $p_z \equiv 1/(\overline{z} - \underline{z})$. The default probability is higher the larger the debt issuance d. As for entitlements, as long as $m < 1$, the default probability is higher the larger y^E and lower the larger the degree of entitlement protection m. Under full protection,

$m = 1$, the level of entitlements does not affect the default probability.[5]

4.3 The bond price and debt revenue

Since $R = 1$ by Assumption 1, the arbitrage condition (1) becomes $q = 1 - \delta$: a rise in the default probability translates one-for-one into a fall in the bond price. Because the default probability can be written as a function of $\{d, y^E\}$, the bond price is expressed as

$$q(d, y^E) = 1 - \delta(d, y^E) = 1 - p_z \left[d + (1 - m)y^E\right]. \tag{15}$$

As in Arellano (2008) and related work, the period-1 government is assumed to choose $\{d, y^E\}$ taking into account the effect of its own actions on the bond price through changes in the default probability.

The revenue from debt issuance, $q(d, y^E)d$, is hump-shaped in d: issuing more raises the default probability, and the bond price $q = 1 - \delta$ falls as this is priced in, so revenue rises only over the range where the fall in price does not overwhelm the increase in quantity. This relationship is summarized by the marginal revenue of debt issuance for given y^E,

$$Q_m(d, y^E) \equiv \frac{\partial \left[q(d, y^E)d\right]}{\partial d} = q(d, y^E) + \frac{\partial q(d, y^E)}{\partial d} \cdot d. \tag{16}$$

The first term on the right-hand side is the revenue from selling one additional unit, and the second is the loss that the price decline inflicts on all inframarginal units. Its sign is characterized by

$$Q_m(d, y^E) \gtreqless 0 \;\Leftrightarrow\; d \lesseqgtr \bar{d}(y^E) = \frac{1/p_z - (1 - m)y^E}{2}.$$

That is, expanding issuance raises revenue as long as issuance stays below $\bar{d}(y^E)$, and revenue peaks at $\bar{d}(y^E)$. This point is called the *Laffer peak*.

The Laffer peak depends on y^E because what depresses the bond price is not d alone but the fiscal gain the period-2 government obtains from default, $d + (1 - m)y^E$. Entitlements account for $(1-m)y^E$ of this gain, so the larger y^E, the lower the bond price at any given issuance and the smaller the issuance at which revenue peaks. Under full protection, $m = 1$, entitlements contribute nothing to this gain and $\bar{d}$ does not depend on y^E. As shown below, equilibrium debt issuance always lies to the left of this peak, below $\bar{d}(y^E)$.

[5] To be precise, default should be defined to occur only when $d > 0$ and $V_S^d(y^E) > V_S^r(z_2, d, y^E)$: when the government holds positive assets ($d < 0$) there is no debt to repudiate and the option of default does not exist, a restriction that (14) does not make explicit. This issue does not arise below, however, because the analysis focuses on the region where equilibrium debt issuance is positive.

4.4 The period-1 government and household problem

The analysis below proceeds under the assumption that $\{d, y^E\}$ satisfy $\delta(d, y^E) < 1$ and lie in the range where consumption and public goods provision are positive both in and out of default. This range is formalized as the feasible set Ω in Section 4.5.1, and Proposition 1 verifies that the equilibrium belongs to it.

The period-1 government and households face two sources of uncertainty: (i) whether the period-2 government defaults, and (ii) the level of period-2 public goods provision. The period-1 government acts taking into account the effect of its own choices on the default probability $\delta(d, y^E)$ and the bond price $q(d, y^E)$. From the preceding discussion, the issuing generation's old-age consumption is $c^{o*}_{2,d} = my^E$ in default and $c^{o*}_{2,r} = y^E$ otherwise, and period-2 public goods provision is given by (5) and (9). With these in mind, the period-1 government and household problem is written as

$$\begin{aligned}
\max_{c_1^y, c_1^o, d, g_1, y^E} \; V_I = \ln c_1^y + \alpha \ln g_1 \\
&+ \delta(d, y^E)\left[\ln(my^E) + \alpha \ln g^*_{2,d}(y^E)\right] \\
&+ (1 - \delta(d, y^E))\Big\{\ln y^E + \alpha E_{z_2}\left[\ln g^*_{2,r}(z_2, d, y^E) \;\middle|\; z_2 \geq \hat{z}(d, y^E)\right]\Big\} \qquad (17) \\
\text{s.t. } c_1^y + c_1^o + g_1 = z_1 + q(d, y^E)d,
\end{aligned}$$

where V_I is the issuing generation's lifetime utility. The default probability δ and the bond price q are given by (14) and (15), the default threshold $\hat{z}$ by (13), and period-2 public goods provision $g^*_{2,r}$ and $g^*_{2,d}$ by (5) and (9). All of these depend on the period-1 government's choices. Because income is fixed at $\underline{z}$ in default, no conditional expectation appears in the term for period-2 public goods provision under default.

Since the initial old's utility does not enter the issuing generation's objective, the transfer to the initial old y_1^o is a pure outflow of resources, and $y_1^{o*} = c_1^{o*} = 0$ holds in equilibrium. Consider now the first-order conditions with respect to $\{d, g_1, y^E\}$ (all derivations are given in Appendix A).

First, the condition for g_1 together with the period-1 resource constraint pins down period-1 consumption and public goods as

$$g_1 = A \cdot \left(z_1 + q(d, y^E)d\right), \quad c_1^y = (1 - A) \cdot \left(z_1 + q(d, y^E)d\right). \qquad (18)$$

Substituting these into the conditions for d and y^E and rearranging, an interior solution (d^*, y^{E*}) is characterized by the following system of two equations, the first-order condition for d, $F = 0$, and that for y^E, $G = 0$:

$$F(d, y^E) = \underbrace{\frac{(1+\alpha)Q_m(d, y^E)}{z_1 + q(d, y^E)d}}_{(\#F1)} + \underbrace{p_z \ln m}_{(\#F2)} + \underbrace{\alpha p_z \ln \frac{X_d(y^E)}{X_r(\overline{z}, d, y^E)}}_{(\#F3)} = 0, \tag{19}$$

$$\begin{aligned} G(d, y^E) = & \underbrace{-\frac{(1+\alpha)(1-m)p_z d}{z_1 + q(d, y^E)d}}_{(\#G1)} + \underbrace{(1-m)p_z \ln m}_{(\#G2)} + \underbrace{\frac{1}{y^E}}_{(\#G3)} \\ & \underbrace{-\frac{\delta(d, y^E)\alpha m}{X_d(y^E)}}_{(\#G4)} + \underbrace{\alpha p_z \ln \frac{X_d(y^E)}{X_r(\overline{z}, d, y^E)}}_{(\#G5)} = 0. \end{aligned} \tag{20}$$

The terms are interpreted as follows. $(\#F1)$ and $(\#G1)$ capture the change in resources available in period 1: expanding d brings in the marginal revenue Q_m, while expanding y^E depresses the bond price and reduces issuance revenue. $(\#F2)$ and $(\#G2)$ capture the effect that expanding d or y^E raises the default probability and thereby the likelihood of the state in which the utility of the issuing generation's own old-age consumption is lower by $\ln m < 0$.

$(\#G3)$ is the increment to old-age utility from one more unit of entitlements, that is, the marginal utility with respect to y^E. Under log utility, the proportional cut in default lowers the utility of old-age consumption only by the constant $\ln m$. The marginal utility with respect to y^E therefore equals $1/y^E$ both in and out of default, and their weighted average $(\#G3)$ depends on neither the degree of protection m nor the default probability. The implication of this property is that m does not affect the choice of entitlements through the marginal utility of old age.[6]

Term $(\#G4)$ is the effect that entitlements squeeze fiscal space in default, X_d, and reduce public goods provision in default. No corresponding term appears in F because in default the debt is wiped out entirely and X_d does not depend on d.

The remaining terms, $(\#F3)$ and $(\#G5)$, capture the change in the expected utility of public goods out of default. Period-2 public goods are provided in proportion to fiscal space. If default occurs, fiscal space is fixed at $X_d(y^E)$; if not, z_2 takes some value in $[\hat{z}, \overline{z}]$, and its realization determines fiscal space $X_r(z_2, d, y^E)$ and hence public goods provision. Expanding d or y^E reduces X_r equally in every non-default state, and these terms aggregate that effect over $z_2 \in [\hat{z}, \overline{z}]$.[7] These two terms take the

[6] This property is specific to log utility. Under CRRA utility $u(c) = c^{1-\sigma}/(1-\sigma)$, the marginal utility with respect to y^E in default is $m^{1-\sigma}(y^E)^{-\sigma}$, which depends on the degree of protection m. If $\sigma > 1$, a fall in m raises this marginal utility, generating an incentive to promise larger entitlements the weaker the protection. This force, however, works in the same direction as this paper's result that y^{E*} is decreasing in m. Hence adding this channel under CRRA utility would not overturn the conclusions that follow.

[7] The lower limit of integration is the threshold $\hat{z}$, where $X_r(\hat{z}(d, y^E), d, y^E) = X_d(y^E)$ holds. $X_d(y^E)$ and $X_r(\overline{z}, d, y^E)$

same form in F and G because d and y^E each reduce non-default fiscal space by the same amount. The economic role of the terms (#$F1$)–(#$G5$) is organized into three effects in Section 4.6.

One property of the equilibrium follows immediately from (19). When $\delta < 1$, that is, when $\hat{z}(d, y^E) < \bar{z}$, we have $X_r(\bar{z}, d, y^E) > X_d(y^E)$, so (#$F3$) is negative; and since $m \leq 1$, (#$F2$) is non-positive. For $F = 0$ to hold, (#$F1$) must therefore be positive, so in equilibrium $Q_m(d^*, y^{E*}) > 0$, that is, $d^* < \bar{d}(y^{E*})$. The issuing generation never borrows beyond the revenue-maximizing point: short of the Laffer peak, additional issuance has the offsetting effects of raising revenue while raising the default probability, but beyond the peak even revenue falls, leaving no room for such borrowing to be chosen.

For an interior point $\{d^*, y^{E*}\}$ satisfying the first-order conditions $F(d, y^E) = 0$ and $G(d, y^E) = 0$ to be a local maximum of the objective V_I, second-order conditions are required. That they are satisfied at the equilibrium point is shown in Step 3 of Appendix B (Lemma 3). Whether a point satisfying the first-order conditions actually exists is established in Section 4.5.2.

4.5 The feasible set and existence of equilibrium

4.5.1 The feasible set

As noted above, this paper focuses on equilibria in which $\{d, y^E\}$ satisfy $\delta(d, y^E) < 1$ and keep consumption and public goods provision positive both in and out of default. Adding non-negativity of debt issuance and entitlements, the feasible set Ω is defined as

$$\Omega \equiv \big\{(d, y^E) \ : \ d \geq 0, \quad y^E \geq 0, \quad \delta(d, y^E) < 1, \quad X_d(y^E) > 0\big\}. \tag{21}$$

The meaning of each condition defining Ω is as follows.

First, the non-negativity constraints $d \geq 0$ and $y^E \geq 0$. Since old-age consumption c_2^o is financed solely by entitlements, the non-negativity of consumption imposes $y^E \geq 0$. Moreover, because this paper focuses on sovereign default, situations in which the government is a net creditor and no default incentive exists are outside the scope of the analysis; attention is therefore restricted to equilibria with $d \geq 0$.

Second, the condition that the default probability stay below one, $\delta(d, y^E) < 1$. From (14), the

appear in these terms because they correspond to the lower and upper limits of integration, respectively. Expanding d or y^E also pushes up the threshold $\hat{z}$ itself, but since fiscal space in the two states coincides at the threshold, this marginal effect cancels out.

condition for the default probability to be interior, $\delta(d, y^E) \in (0,1)$, is

$$\hat{z}(d, y^E) \in (\underline{z},\ \overline{z}) \ \Leftrightarrow\ 0 < d + (1-m)y^E < 1/p_z.$$

The lower bound requires that default risk not vanish, and the upper bound that default not occur with probability one. Only the upper bound is imposed in Ω, namely

$$d < 1/p_z - (1-m)y^E \equiv d_h(y^E). \tag{22}$$

The upper bound is needed because when default occurs with probability one, the conditional expectation over non-default states is undefined and the bond price falls to zero, so issuance yields no revenue.

The lower bound $\delta > 0$, by contrast, is not included in the definition of Ω for the following reason. As shown later, $d = y^E = 0$ never arises in equilibrium, and $d^* > 0$ when $m = 1$. Hence $d + (1-m)y^E > 0$, that is, $\delta > 0$, always holds in equilibrium, and a situation in which default never occurs cannot arise. Since the lower bound is a consequence of equilibrium rather than an assumption, the only binding part of the feasible set is the upper bound $\delta < 1$.

The analysis therefore targets equilibria in which the default probability is interior. The reason for focusing on such equilibria is that, since the paper's subject is how changes in the degree of protection m move default risk, there is simply nothing to move in situations where default never occurs ($\delta^* = 0$) or always occurs ($\delta^* = 1$).

Third, the condition that fiscal space in default be positive, $X_d(y^E) > 0$. When the period-2 government chooses default, the successor generation's consumption and public goods provision are, from (9), $c^{y*}_{2,d} = (1-A)X_d(y^E)$ and $g^*_{2,d} = A \cdot X_d(y^E)$. For these to be positive, fiscal space in default, $X_d(y^E) = \underline{z} - my^E$, must be positive; that is,

$$X_d(y^E) > 0 \ \Leftrightarrow\ y^E < \frac{\underline{z}}{m} \equiv y^E_h. \tag{23}$$

This condition means that entitlements y^E are not excessive—more concretely, that the amount guaranteed even in default, my^E, falls short of the lower bound of period-2 income $\underline{z}$.

Noting that default is not chosen when $z_2 \geq \hat{z}(d, y^E)$, fiscal space in the non-default case satisfies

$$X_r(z_2, d, y^E) \geq X_r(\hat{z}(d, y^E), d, y^E) = \underline{z} - my^E = X_d(y^E).$$

Hence, if condition (23) holds, fiscal space in the non-default case is positive as well.

Collecting the three conditions, the feasible set Ω can also be written, using $y_h^E = \underline{z}/m$ and $d_h(y^E)$, as

$$\Omega = \{(d, y^E) \ : \ 0 \le y^E < y_h^E, \ \ 0 \le d < d_h(y^E)\}.$$

The equilibria analyzed in this paper are those belonging to this set Ω.

4.5.2 Existence of equilibrium

With these preparations, we show that a pair $\{d, y^E\}$ satisfying the first-order conditions (19) and (20) exists within the feasible set Ω.

Proposition 1. *Suppose Assumption 1 holds and z_1 is sufficiently small. Then there exists a lower bound $\underline{m} \in (0,1)$ such that for each $m \in [\underline{m}, 1]$, at least one equilibrium $(d^*, y^{E*}) \in \Omega$ with default probability $0 < \delta^* < 1$ exists.*

See Appendix B for the proof.

The only condition the proposition requires is that period-1 income z_1 be sufficiently small, that is, that the issuing generation have a sufficiently strong borrowing motive to extract resources from the future. The lower bound $\underline{m}$ defined by the proposition is called the *borrowing shutdown level.* When the degree of protection lies in $m \in (\underline{m}, 1]$, equilibrium debt issuance is positive ($d^* > 0$), and at the left endpoint $m = \underline{m}$, $d^* = 0$. The borrowing shutdown level also depends on α; below, it is written $\underline{m}(\alpha)$ only where the dependence on α needs to be made explicit.

The core implication of the proposition is that the existence of equilibrium ultimately reduces to a single question: whether the issuing generation chooses positive borrowing. Because the issuing generation receives the guaranteed portion my^E as the old in period 2, default damages its own old-age consumption. The weaker the protection m, the larger this self-inflicted cost, and the less it pays to undertake borrowing that raises the default probability. Likewise, the higher period-1 income z_1, the weaker the incentive to borrow against the future. Taking z_1 small and taking m large thus do not serve separate constraints: both work to strengthen the same borrowing motive, and both support positive debt issuance. When protection weakens to $\underline{m}$, this borrowing motive is exactly extinguished and equilibrium issuance is $d^* = 0$. Because z_1 and $\underline{m}$ operate as substitutes in this way, the smaller z_1, the lower the borrowing shutdown level $\underline{m}$ and the wider the analysis range $[\underline{m}, 1]$.

Non-negativity of entitlements, $y^E \geq 0$, is always satisfied at an interior point regardless of m or z_1, because under log utility marginal utility diverges as old-age consumption approaches zero. What delimits existence is the non-negativity of debt issuance, and it is this that defines the borrowing shutdown level $\underline{m}$. Furthermore, that the equilibrium default probability stays interior, $\delta^* \in (0, 1)$, is guaranteed as a consequence of optimization rather than an additional assumption. As certain default ($\delta \to 1$) is approached, the bond price $q = 1 - \delta$ approaches zero, so additional issuance no longer generates revenue, while the issuing generation's own old-age guarantee my^E is almost surely damaged. The marginal value of borrowing necessarily turns negative before the certain-default boundary, so the issuing generation never voluntarily borrows its way into certain default, and $\delta^* < 1$ holds in equilibrium.

The comparative statics with respect to m that follow (Sections 5 and 6) are conducted over the range $m \in [\underline{m}, 1]$, where debt issuance is non-negative ($d^* \geq 0$).

4.6 Three effects shaping the equilibrium

The forces determining equilibrium debt issuance d^* and entitlement setting y^{E*} can be organized into three effects. The first two are the crowd-out effect and the smoothing effect identified by Bouton et al. (2020), both of which concern the allocation of period-2 public goods. The third is the default-risk effect, which stems from the fact that extraction moves the default probability and appears only once default is endogenized, as in this paper. The three effects are defined in turn below, and the impact of changes in the degree of protection m on the equilibrium quantities through each effect is then laid out. The implications of the comparative statics in subsequent sections can be interpreted through this organization.

The solution to the period-2 problem, whether or not default occurs, was written using period-2 fiscal space X ($X_r = z_2 - d - y^E$ out of default, $X_d = \underline{z} - my^E$ in default) as $c_2^{y*} = (1 - A)X$ and $g_2^* = AX$, where $A = \alpha/(1 + \alpha)$. That is, of each unit of fiscal space X left to period 2, only A units go to public goods g_2, and the remaining $1 - A$ units go to the successor generation's private consumption c_2^y.

The crowd-out effect

Even if the issuing generation restrains d and y^E and leaves resources to the future, only A units of them become public goods; the remaining $1 - A = 1/(1 + \alpha)$ is absorbed into the successor generation's private consumption. Conversely, when the issuing generation expands d or y^E by one unit and extracts resources from the future, the period-2 public goods lost in exchange amount to only A units. Since the issuing generation can capture the extracted resources as period-1 consumption, period-1 public goods, or its own old-age entitlements, while the reward for restraint (the increment in public goods) is only A, leaving resources to the future involves a "leak."

This leak gives the issuing generation an incentive to expand d and y^E. This is called the *crowd-out effect*. Its size is measured by the leakage share $1 - A = 1/(1+\alpha)$ and is the same for debt and entitlements. The effect is unchanged from Bouton et al. (2020), because the allocation shares of period-2 fiscal space are the same whether or not default occurs.

The smoothing effect

Following Bouton et al. (2020), this effect stems from the incentive to smooth public goods provision across the two periods. Expanding d or y^E to extract resources from the future shrinks period-2 fiscal space X and reduces $g_2 = AX$. This enriches period-1 resources at the expense of period-2 public goods provision, which is undesirable for an issuing generation that prefers smoothing. Internalizing this cost—the fall in g_2—generates an incentive to restrain the expansion of d and y^E. This is called the *smoothing effect*.

The size of the smoothing effect is measured by the smoothing cost, that is, the expected loss of period-2 public goods from expanding d or y^E by one unit. Using the period-2 solution $g_2^* = AX$, the state-by-state marginal effects are

$$\frac{\partial g_2^*}{\partial d} = \begin{cases} 0 & \text{with default} \\ -A & \text{otherwise} \end{cases} \qquad \frac{\partial g_2^*}{\partial y^E} = \begin{cases} -mA & \text{with default} \\ -A & \text{otherwise} \end{cases}$$

Debt is not repaid in default and thus does not cut period-2 public goods there, whereas the protected portion of entitlements, my^E, is paid even in default, squeezing fiscal space by m per unit of y^E.

Given that default occurs with probability δ, the smoothing costs are[8]

$$-\mathbb{E}\left[\frac{\partial g_2^*}{\partial d}\right] = (1-\delta)A, \quad -\mathbb{E}\left[\frac{\partial g_2^*}{\partial y^E}\right] = \delta m A + (1-\delta)A. \tag{24}$$

The smoothing cost thus splits into a non-default part, $(1-\delta)A$, which arises for both instruments, and a default part, δmA, which arises only for entitlements. The two instruments differ only in the

[8] At the boundary $\hat{z}$, $X_r\big(\hat{z}(d, y^E), d, y^E\big) = X_d(y^E)$ holds; for the marginal probability mass exiting the default state, the public goods levels in the two states coincide, so the terms operating through changes in $\hat{z}$ cancel out.

latter: entitlements are more costly by $\delta m A > 0$. Debt lacks this part because in default the debt is wiped out entirely and does not cut the resources available in default. The non-default part corresponds to the smoothing effect in Bouton et al. (2020); the default part appears only in this paper, where sovereign default is endogenized.

Importantly, the burden this cost generates is not determined by the quantity of public goods lost alone. Period-2 public goods provision in default is smaller than under any income realization, because income falls to its lower bound and guaranteed entitlements must still be paid. Since marginal utility is diminishing, it is precisely in this state of lowest provision that losing one unit of public goods hurts the most. Moreover, the larger the entitlements, the smaller public goods provision in default becomes, and the heavier the burden of the same one-unit loss.

The default-risk effect

Because the issuing generation receives entitlements as the old in period 2, the fact that its own extraction raises the default probability $\delta = p_z[d + (1-m)y^E]$ carries a cost that rebounds on itself. Expanding d or y^E pushes up δ, which harms the issuing generation through two channels. First, it lowers the bond price $q = 1 - \delta$, making it harder to raise period-1 resources through debt issuance. Second, it raises the frequency with which its own old-age entitlements are cut to the guaranteed portion my^E, which happens with probability δ. Internalizing these two costs generates an incentive to restrain the expansion of d and y^E. This is called the *default-risk effect*.

Its size is proportional to the degree to which each instrument moves the default probability. From $\delta = p_z\left[d + (1-m)y^E\right]$,

$$\frac{\partial \delta}{\partial d} = p_z, \quad \frac{\partial \delta}{\partial y^E} = (1-m)p_z.$$

Entitlements move the default probability less, by the factor $(1-m)$, because in default the debt is wiped out entirely while part of the entitlements remains guaranteed and only the fraction $1-m$ is cut. The fiscal outlay saved by default is correspondingly smaller, so entitlements do less to strengthen the period-2 government's incentive to default. This effect does not appear in Bouton et al. (2020); it is specific to this paper, which endogenizes default.

How stronger protection affects the equilibrium quantities

The impact of a rise in the degree of protection m on debt issuance and entitlement setting can be organized through the three effects. Consider first the crowd-out effect. Its size is determined by the allocation share A alone and does not involve the degree of protection, so changes in m leave it unaffected.

Consider next the smoothing effect. By (24), holding the default probability fixed, the larger m, the larger the smoothing cost of entitlements: the guaranteed portion is paid even in default and cuts period-2 public goods by that much more. A rise in m therefore strengthens the force restraining entitlements. Since m does not appear in the smoothing cost of debt, this channel operates on entitlements only.[9]

Finally, m operates on the default-risk effect in two ways. First, the larger m, the larger the entitlements guaranteed in default and the smaller the disutility suffered from the cut. This weakens the force restraining extraction through both instruments and expands both. Second, the larger m, the weaker the effect of entitlements on the default probability: with a larger share of entitlements guaranteed, the period-2 government saves less by defaulting, and default becomes correspondingly less attractive. This too weakens the force restraining entitlements and promotes their expansion.

We now summarize these effects for each instrument. For debt, the only effect that m moves is the default-risk effect, which weakens; stronger protection therefore raises equilibrium debt issuance d^*. For entitlements, the smoothing effect tightens the restraint while the default-risk effect loosens it, so the direction of equilibrium entitlement setting y^{E*} depends on their relative sizes and is in general indeterminate.

When α is small, the matter is settled. Because the smoothing effect restraining extraction is weak relative to the crowd-out effect promoting it, the issuing generation sets both d and y^E high, squeezing period-2 public goods provision. In default especially, income falls to its lower bound and the guaranteed entitlements must still be paid, so public goods provision is at its smallest. In this situation the smoothing cost responds sharply to m: even a slight fall in provision raises marginal utility sharply. Stronger protection therefore pushes the smoothing cost up steeply, and the force restraining entitlements operates powerfully. The loosening of the default-risk effect, by contrast, is finite and mild and cannot offset it. Hence, if α is sufficiently small, y^{E*} is decreasing in m.

Figure 4 plots the equilibrium quantities as m moves from the borrowing shutdown level $\underline{m} \simeq 0.30$ to 1 under $\underline{z} = 1$, $\overline{z} = 6.5$, $z_1 = 2.5$, $\alpha = 0.03$. Since $R = \beta = 1$ by Assumption 1 and p_z is determined by $(\underline{z}, \overline{z})$, these four parameters exhaust the exogenous parameters. This configuration is hereafter

[9] The entire effect of m on entitlement setting operates from this cost side. Under log utility, old-age utility in default decomposes as $\ln(my^E) = \ln m + \ln y^E$, and the marginal utility of y^E is $1/y^E$, independent of the degree of protection ((#$G3$) in (20)). Hence the benefit-side channel—"the weaker the protection, the larger the entitlement set to secure the same old-age consumption"—is exactly zero in this model. This is a property specific to log utility; under CRRA utility ($\sigma > 1$) this channel appears, but it works in the direction of making y^{E*} decreasing in m, that is, in the same direction as the results obtained below, so the main conclusions are unchanged (footnote 6).

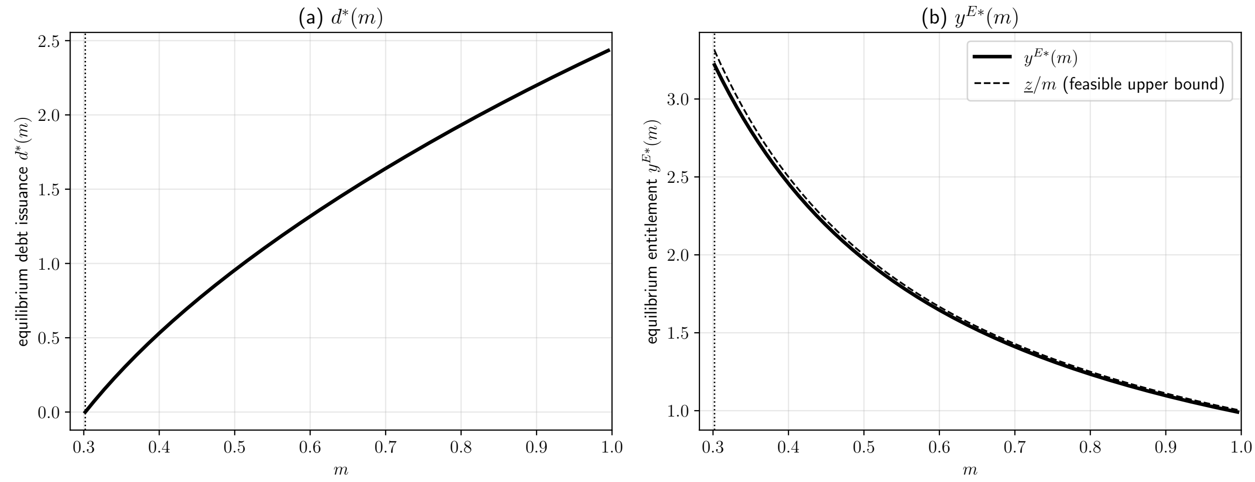

Figure 4: Equilibrium quantities in the baseline ($\underline{z} = 1,\ \overline{z} = 6.5,\ z_1 = 2.5,\ \alpha = 0.03$). The vertical dotted line marks the borrowing shutdown level $\underline{m} \simeq 0.30$. Panel (a): equilibrium debt issuance $d^*(m)$. Panel (b): equilibrium entitlement setting $y^{E*}(m)$ (the dashed line is the level $\underline{z}/m$ that exhausts the resources available in default).

called the baseline, and all subsequent numerical examples take it as the reference point.

At the left endpoint $m = \underline{m}$, $d^* = 0$, and d^* increases monotonically in m. By contrast, y^{E*} is decreasing in m and, under the small α of this parameterization, tracks close to the level at which public goods provision in default is nearly exhausted (the dashed line in the figure). The decline of y^{E*} is preserved even when α is taken large enough that the equilibrium is clearly interior (Figure 10 in Appendix G).

The comparative statics with respect to m in the following sections build on this relationship: stronger protection raises debt issuance and reduces entitlement setting.

5 The effect of entitlement protection on the default probability

This section analyzes the effect of the degree of entitlement protection m on the equilibrium default probability. The degree of protection operates on the default probability through three channels: through equilibrium debt issuance, through equilibrium entitlement setting, and directly, even holding both fixed. The two propositions in this section show that, under certain conditions, the sign of the combined overall effect reverses with the level of protection: where protection is very weak, strengthening it lowers the default probability; near full protection, strengthening it raises the probability. From the standpoint of minimizing the default probability, this means that entitlement protection should be neither too strong nor too weak.

5.1 Decomposing the default-probability effects of m

To make explicit the degree of protection m, the object of the comparative statics, the equilibrium quantities are written as functions of m: $d^*(m), y^{E*}(m)$. From (15), the equilibrium default probability is

$$\delta^*(m) \;=\; p_z\big[d^*(m) + (1-m)y^{E*}(m)\big].$$

The bracketed term $d + (1-m)y^E$ is the fiscal outlay saved when the period-2 government chooses default: default extinguishes the repayment obligation d and, in addition, makes available the portion of entitlements flowing into the general budget, $(1-m)y^E$. The larger this saving, the more advantageous default is for the period-2 government, and the higher the default probability.

Totally differentiating with respect to m decomposes the effect into the following three terms.

$$\frac{d\delta^*(m)}{dm} \;=\; p_z\Bigg[\underbrace{\frac{dd^*(m)}{dm}}_{\text{borrowing response}} + \underbrace{(1-m)\frac{dy^{E*}(m)}{dm}}_{\text{entitlement response}} \underbrace{-\, y^{E*}(m)}_{\text{direct effect}}\Bigg]. \tag{25}$$

The first term is the increase due to stronger protection raising debt issuance (the *borrowing response*). The second is the decrease due to stronger protection reducing entitlement setting (the *entitlement response*), and the third is the decrease that arises, even holding both fixed, because the share flowing into the general budget in default, $(1-m)$, itself shrinks (the *direct effect*). When the utility weight on public goods α is sufficiently small, d^* is increasing and y^{E*} decreasing in m, as seen in the previous section, so the first term is positive and the second and third are negative. The behavior of $\delta^*(m)$ is therefore determined by whether the increase on the debt side or the decrease on the entitlement side dominates. The two propositions below show that which side dominates reverses with the level of protection.

5.2 How stronger protection lowers default risk near the borrowing shutdown level

Consider first the lower end of the analysis range, that is, the neighborhood of the borrowing shutdown level $m = \underline{m}$ at which equilibrium debt issuance is exactly zero. The following proposition holds.

Proposition 2. *Suppose Assumption 1 holds and z_1 is small enough that the borrowing shutdown level $\underline{m} \in (0,1)$ (Proposition 1) is defined. Suppose further that the $\alpha \to 0$ limit of the borrowing shutdown* *level, $\underline{m}(0)$, satisfies the following two conditions.*

$$\textit{(CP2a)} \quad p_z\big(1-\underline{m}(0)\big)\frac{\underline{z}}{\underline{m}(0)}\big(-\ln \underline{m}(0)\big) \leq 1,$$

$$\textit{(CP2b)} \quad z_1\Big(\underline{m}(0) - p_z\underline{z}\big(\ln \underline{m}(0)\big)^2\Big) \leq \underline{z}.$$

Then, if α is sufficiently small, in a neighborhood of the borrowing shutdown level $\underline{m}$,

$$\frac{d\delta^*(m)}{dm} < 0.$$

That is, a marginal strengthening of entitlement protection near $m = \underline{m}$ lowers the equilibrium default probability.

See Appendix C for the proof.

First, consider what the conditions of the proposition require. Conditions (CP2a) and (CP2b) are imposed not on the borrowing shutdown level itself but on its limit $\underline{m}(0)$ because the proposition is proved using the $\alpha \to 0$ limit (Appendix C). Since $\underline{m}(0)$ is uniquely determined by the exogenous parameters alone, the two conditions are effectively conditions on exogenous parameters. That "z_1 is sufficiently small" guarantees, among other things, that an equilibrium with $\delta^* \in (0,1)$ exists at every point of $m \in [\underline{m}, 1]$.[10] That "α is sufficiently small" means the crowd-out effect operates strongly and the smoothing effect weakly, and condition (CP2a) means the default-risk effect on entitlements does not operate too strongly. Together they imply that entitlements are set high and y^{E*} is decreasing in m (Section 4.6). The role of condition (CP2b) is described at the end of the discussion below.

Strengthening protection produces two effects through entitlements. First, the entitlements that are cut in default and flow into the general budget shrink (the direct effect). Second, stronger protection raises the default-state smoothing cost of entitlements, so entitlement setting itself falls (the entitlement response). Both reduce the fiscal outlay saved by default, weakening the period-2 government's default incentive.

On the debt side, the opposite happens. Holding debt issuance fixed, the default probability falls by the amount of the direct effect and the entitlement response, and the bond price rises. This makes raising period-1 resources more favorable, so the issuing generation increases issuance (the borrowing response). In addition, as protection strengthens, the disutility from entitlements being cut when default occurs shrinks, making the issuing generation more willing to accept a higher frequency of

[10]See Appendix B and Appendix G for details.

default from increased issuance—another reason debt issuance rises. Larger issuance increases the repayment saved by default and thus strengthens the period-2 government's default incentive.

The impact of a rise in m on the default probability is therefore determined by whether the restraint operating through entitlements or the stimulus operating through debt dominates. In this region of weak entitlement protection, the former is large: entitlements are set high because α is small, and, protection being weak, the share flowing into the general budget in default is also large.

What determines the size of the borrowing response is the resources available in period 1, z_1. In equilibrium, the marginal benefit of additional debt issuance balances against the marginal cost of the higher default probability it brings. When stronger protection shrinks the latter, issuance rises to restore the balance. But when period-1 resources are scarce, the benefit of an additional unit of resources is large to begin with, so only a small increase in issuance restores the balance. Condition (CP2b) requires that period-1 resources z_1 be small relative to the resources available in default. Under this condition the borrowing response stays small, and the restraint on the entitlement side dominates the stimulus on the debt side.

5.3 How weaker protection lowers default risk near full protection

Consider next the upper end of the analysis range, that is, the neighborhood of full protection, $m = 1$. The following proposition holds.

Proposition 3. *Suppose Assumption 1 holds and z_1 is small enough that the borrowing shutdown level $\underline{m} \in (0, 1)$ (Proposition 1) is defined. Suppose further that z_1 satisfies the following condition.*

$$\textit{(CP3)} \quad z_1 > \frac{5\underline{z} - \overline{z}}{4}.$$

Then, if α is sufficiently small, in a neighborhood of full protection $m = 1$,

$$\frac{d\delta^*(m)}{dm} > 0.$$

That is, a marginal relaxation of entitlement protection near $m = 1$ lowers the equilibrium default probability.

See Appendix D for the proof.

First, as in Proposition 2, that "z_1 is sufficiently small" guarantees, among other things, that an equilibrium with $\delta^* \in (0, 1)$ exists at every point of $m \in [\underline{m}, 1]$. Second, near full protection, the condition that “α is sufficiently small” alone guarantees that entitlements are set high and y^{E*} is decreasing in m (Section 4.6). The role of condition (CP3) is described at the end of the discussion below.

As in Proposition 2, stronger protection brings both restraint through entitlements and stimulus through debt, and the outcome is decided by which dominates. In the region close to full protection, however, their relative sizes are reversed compared to the neighborhood of $m = \underline{m}$.

The former is small. Stronger protection shrinks the entitlements flowing into the general budget in default (the direct effect), but in this region entitlement setting itself is small, so the effect is small. And although stronger protection reduces the setting further, the share flowing into the general budget in default is nearly zero, so the fiscal outlay saved barely moves: the entitlement response has all but vanished.

The latter, by contrast, is large. Near full protection, entitlements are paid almost in full even when default occurs. Since almost no disutility from entitlement cuts remains, the issuing generation willingly accepts a higher frequency of default and issues large amounts of debt. That the size of the borrowing response depends on period-1 resources was seen in Proposition 2, and condition (CP3) requires that period-1 resources not be too small. In particular, in economies where the upper bound of period-2 income is sufficiently large relative to the lower bound ($\overline{z}/\underline{z} > 5$), the condition is satisfied automatically for any $z_1 > 0$: the wider the income range, the smaller the rise in the default probability from one additional unit of debt, and the larger borrowing tends to be. The borrowing response is then large, and the stimulus on the debt side dominates the restraint on the entitlement side.

5.4 How the degree of protection shifts the composition of obligations

Taken together, the two propositions imply that $\delta^*(m)$ slopes downward near the borrowing shutdown level and upward near full protection. That is, where entitlement protection is weak, strengthening it lowers the equilibrium default probability, and where it is strong, relaxing it does. From the standpoint of minimizing default risk, entitlement protection should be neither too strong nor too weak. This subsection lays out where this non-monotonicity comes from.

What determines the period-2 government's incentive to default is the total fiscal outlay saved by defaulting, $d^* + (1 - m)y^{E*}$. Debt is written off in full, whereas only the unprotected fraction $1 - m$ of entitlements is saved. This asymmetry gives the degree of protection m a special role: m governs which of the two instruments is the cheaper means for the issuing generation of extracting resources from the future, and by how much.

When protection is weak, most of the promised entitlements are cut in default. Adding one unit to the promise obliges actual payment of only m units in default, weighing little on period-2 resources. Extraction through entitlements is cheap in this sense, and the issuing generation sets them high. Debt issuance, by contrast, is restrained: weak protection also means that one's own old-age share is deeply cut when default occurs, so borrowing that raises the default probability does not pay. Extraction thus tilts toward entitlements, most of which become fiscal outlays the period-2 government can save.

When protection is strong, the situation reverses. Promised entitlements are honored almost in full even in default, so increasing the promise amounts to increasing the obligation to pay in default. Scarce resources go to entitlement payments and public goods provision shrinks, so the issuing generation restrains entitlements. For the same reason, however, debt issuance is encouraged: since the pre-set entitlements are barely cut even when default occurs, the pain of a higher default probability is small. Extraction thus tilts toward debt.

Strengthening protection thus shifts extraction from entitlements to debt. What m moves is the composition of the instruments through which the issuing generation extracts resources from the future. Because protection that is too weak inflates entitlements and protection that is too strong inflates debt, extraction concentrates on one instrument at either extreme. Under weak protection, large entitlements are set, and the fiscal outlay that can be saved in default, $(1-m)y^{E*}$, is large. Under strong protection, the debt that is fully written off in default, d^*, is large.

Viewed differently, each instrument carries its own check borne by the issuing generation itself, and at each extreme one of the checks is released. What restrains entitlements is that the guaranteed portion my^{E*} must be paid even in default, diverting scarce resources away from public goods—a check that shrinks as protection weakens. What restrains debt is that a higher default probability cuts one's own old-age share $(1-m)y^{E*}$—a check that shrinks as protection strengthens. At intermediate degrees of protection both checks operate simultaneously, so neither instrument is used to the extreme, and the total fiscal outlay that can be saved in default, $d^* + (1-m)y^{E*}$, stays small.

Consequently, whether protection is too weak or too strong, the period-2 government's default incentive is strong and the equilibrium default probability high. This is why the degree of protection that minimizes default risk lies at neither extreme but strictly between them. The policy implications of this result in light of real-world institutions are discussed in Section 8.

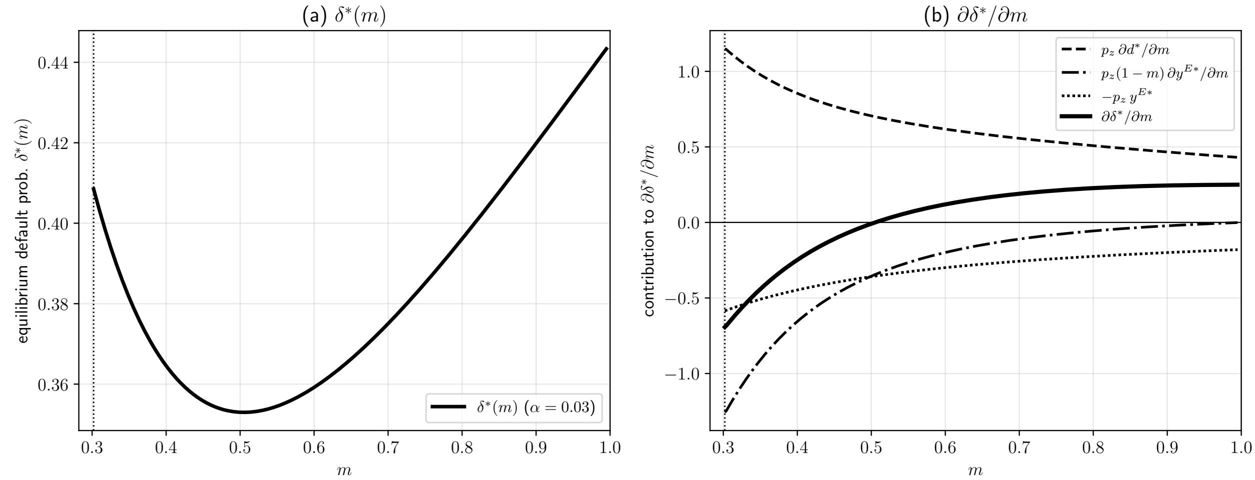

Figure 5: The equilibrium default probability and its decomposition (baseline: $\underline{z} = 1,\ \overline{z} = 6.5,\ z_1 = 2.5,\ \alpha = 0.03$). The vertical dotted line marks the borrowing shutdown level $\underline{m} \simeq 0.30$. Panel (a): $\delta^*(m)$. Panel (b): the three terms of (25)—the borrowing response (black dashed), the entitlement response (gray dash-dotted), the direct effect (gray dotted), and their sum (thick black solid).

5.5 A numerical illustration of Propositions 2 and 3

The results above are now confirmed numerically under the same baseline as in Section 4.6, $\underline{z} = 1,\ \overline{z} = 6.5,\ z_1 = 2.5,\ \alpha = 0.03$. Under this configuration the borrowing shutdown level is $\underline{m} \simeq 0.30$ (in the $\alpha \to 0$ limit, $\underline{m}(0) \simeq 0.29$), and conditions (CP2a) and (CP2b) of Proposition 2 and condition (CP3) of Proposition 3 are all satisfied.

Panel (a) of Figure 5 shows the equilibrium default probability $\delta^*(m)$. It slopes downward near the borrowing shutdown level and upward near full protection, confirming the claims of both propositions. Panel (b) plots the three terms of (25). Near the borrowing shutdown level, the sum of the entitlement response and the direct effect exceeds the borrowing response; near full protection, the entitlement response has all but vanished and the borrowing response dominates. The sign reversal in Propositions 2 and 3 can be seen to arise from the relative sizes of these effects.

Moreover, in this numerical example $\delta^*(m)$ is U-shaped over the whole range, with a unique interior degree of protection that minimizes the default probability. The two propositions assert only the sign of the slope at the two endpoints and do not guarantee the shape in between—in particular, whether the minimum is unique. What the figure shows is that, at least under these parameters, a unique interior minimum exists. The shape of $\delta^*(m)$ also depends on α; this point is revisited in Section 7.

6 The effect of entitlement protection on welfare

Propositions 2 and 3 in the previous section showed that the equilibrium default probability $\delta^*(m)$ falls when entitlement protection is strengthened near the borrowing shutdown level $m \to \underline{m}$, and when it is relaxed near full protection. A lower default probability, however, is not necessarily desirable for every generation: an adjustment that lowers the default probability might merely benefit one generation at the expense of the other.

This section shows that the adjustments of protection identified by Propositions 2 and 3—strengthening protection near the borrowing shutdown level $m = \underline{m}$ and relaxing it near full protection $m = 1$—not only lower the default probability but also improve the welfare of the issuing generation and the successor generation simultaneously; that is, they are Pareto improving. The two propositions in this section pair with Propositions 2 and 3, respectively, and share their parameter conditions.

6.1 Defining the welfare of the two generations

We begin by defining the ex ante welfare of each generation. The issuing generation lives through both periods, so its welfare is the sum of utilities over the two periods; the successor generation exists only in period 2, so its welfare consists of period-2 utility alone. Both are evaluated before period-2 income z_2 is realized, so expectations are taken over z_2.

The realization of z_2 affects the welfare of both generations through two channels. First, z_2 determines whether default occurs: default occurs when $z_2 < \hat{z}(d, y^E)$, and its probability is $\delta(d, y^E)$. Second, when default does not occur, the level of z_2 determines period-2 fiscal space $X_r(z_2, d, y^E)$ and the public goods provision it finances, $g^*_{2,r}(z_2, d, y^E)$. In default, income is fixed at the lower bound $\underline{z}$, so this second channel does not operate.

The issuing generation's welfare V_I

The issuing generation lives as the young in period 1 and the old in period 2. Its ex ante welfare is period-1 (young) utility plus expected period-2 (old) utility, which is none other than the objective (17) maximized by the period-1 government and households. Using (18), the fact that $c_1^{o*} = 0$ in equilibrium, and (A1) in Appendix A, the issuing generation's equilibrium welfare is expressed as

$$\begin{aligned} V_I(d^*, y^{E*}; m) =& (1+\alpha)\ln\bigl(z_1 + q(d^*, y^{E*})d^*\bigr) \\ & + \delta(d^*, y^{E*})\ln m + \ln y^{E*} \end{aligned}$$

$$
\begin{aligned}
&+ \alpha\delta(d^*, y^{E*}) \ln X_d(y^{E*}) \\
&+ \alpha p_z \cdot \Big[X_r(\overline{z}, d^*, y^{E*}) \ln X_r(\overline{z}, d^*, y^{E*}) - X_r(\overline{z}, d^*, y^{E*}) \\
&\qquad\qquad\qquad - X_d(y^{E*}) \ln X_d(y^{E*}) + X_d(y^{E*})\Big] \\
&+ 2\alpha \ln A + \ln(1 - A). \qquad (26)
\end{aligned}
$$

The first term on the right-hand side is the utility from period-1 (young) consumption and public goods. The second and third terms are the expected utility from period-2 (old) entitlement consumption, expressed as non-default consumption $\ln y^{E*}$ plus the loss from the cut to the fraction m in default, $\delta \ln m < 0$. The fourth and fifth terms are the expected utility from period-2 public goods, corresponding to the default and non-default states, respectively. The remaining terms are constants.

Since the equilibrium quantities (d^*, y^{E*}) are both determined by the degree of entitlement protection m, (26) gives the issuing generation's welfare as a function of m alone, hereafter written $V_I^*(m) \equiv V_I\big(d^*(m), y^{E*}(m); m\big)$.

The successor generation's welfare V_S

The successor generation exists only as the young in period 2. Its ex ante welfare is the default-probability-weighted average of the non-default utility V_S^r and the default utility V_S^d,

$$
V_S = (1 - \delta)\mathbb{E}_{z_2}\big[V_S^r(z_2, d, y^E) \mid z_2 \geq \hat{z}\big] + \delta V_S^d(y^E),
$$

where V_S^r and V_S^d are the period-2 indirect utilities derived in Section 4, given by (7) and (11), respectively.

Substituting (7), (11), and (A1) in Appendix A, the equilibrium welfare can be expressed as

$$
\begin{aligned}
V_S(d^*, y^{E*}; m) =& (1 + \alpha)\delta(d^*, y^{E*}) \ln X_d(y^{E*}) \\
&+ (1 + \alpha)p_z \Big[X_r(\overline{z}, d^*, y^{E*}) \ln X_r(\overline{z}, d^*, y^{E*}) - X_r(\overline{z}, d^*, y^{E*}) \\
&\qquad\qquad\qquad - X_d(y^{E*}) \ln X_d(y^{E*}) + X_d(y^{E*})\Big] \\
&+ \alpha \ln A + \ln(1 - A). \qquad (27)
\end{aligned}
$$

The first term on the right-hand side is the expected utility in default, and the second the expected utility out of default; the remaining terms are constants.

As with the issuing generation's welfare, since the equilibrium quantities (d^*, y^{E*}) are determined by the degree of protection m, (27) gives the successor generation's welfare as a function of m alone, hereafter written $V_S^*(m) \equiv V_S\big(d^*(m), y^{E*}(m); m\big)$.

6.2 Decomposing the welfare effects of m

This subsection organizes the channels through which changes in m act on the welfare of the two generations. Consider first the issuing generation. Since the issuing generation optimizes over $\{d, y^E\}$ at an interior point, the envelope theorem implies that the effects operating through changes in the equilibrium policies $\{d^*, y^{E*}\}$ cancel out. This property is specific to the issuing generation, which optimizes over $\{d, y^E\}$, and does not hold for the successor generation. From (26), the effect of a change in the degree of entitlement protection m on the issuing generation's welfare is organized into the following three channels.

$$\frac{dV_I^*}{dm} = \underbrace{(1+\alpha)\frac{p_z y^{E*} d^*}{z_1 + q^* d^*}}_{\text{bond price channel}} \underbrace{-p_z y^{E*} \ln m + \frac{\delta^*}{m}}_{\substack{\text{default-state}\\\text{consumption channel}}} \underbrace{-\frac{\alpha \delta^* y^{E*}}{X_d(y^{E*})}}_{\substack{\text{default-state}\\\text{public goods channel}}}. \tag{28}$$

The first term is the positive effect whereby stronger protection lowers the period-2 default probability, raising the bond price and increasing the resources the issuing generation can raise. The second is the positive effect whereby stronger protection shrinks the entitlement cut in default, raising consumption, while also lowering the default probability and hence the frequency with which the cut is suffered. The third is the negative effect whereby stronger protection increases the entitlements guaranteed in default, reducing public goods provision in default by that amount.

Consider next the successor generation. Unlike the issuing generation, the successor generation does not optimize over $\{d, y^E\}$ but inherits the equilibrium policies as given. The envelope theorem therefore does not apply, and the derivative with respect to m is a total derivative including the indirect effects through changes in the equilibrium policies $\{d^*, y^{E*}\}$. From (27),

$$\begin{aligned}\frac{dV_S^*}{dm} =& (1+\alpha)\Big[\frac{d\delta^*}{dm}\ln X_d(y^{E*}) + \delta\frac{1}{X_d(y^{E*})}\frac{dX_d}{dm} \\ &+ p_z\Big(\frac{dX_r}{dm}\ln X_r(\overline{z}, d^*, y^{E*}) - \frac{dX_d}{dm}\ln X_d(y^{E*})\Big)\Big],\end{aligned} \tag{29}$$

where, from (6), (10), and (25),

$$\frac{d\delta^*}{dm} = p_z\Big(\frac{dX_d}{dm} - \frac{dX_r}{dm}\Big).$$

Substituting this into (29) and rearranging, the effect of the degree of protection m on the successor generation's welfare can be organized as follows.

$$\frac{dV_S^*}{dm} = (1+\alpha)\Big[\underbrace{\Big(\frac{\delta}{X_d(y^{E*})} + p_z \ln\frac{X_r(\overline{z}, d^*, y^{E*})}{X_d(y^{E*})}\Big)\frac{dX_d}{dm}}_{\substack{\text{default-state} \\ \text{fiscal space channel}}} \underbrace{- \ln\frac{X_r(\overline{z}, d^*, y^{E*})}{X_d(y^{E*})} \cdot \frac{d\delta^*}{dm}}_{\text{default probability channel}}\Big]. \tag{30}$$

The first term on the right-hand side represents the effect of stronger protection changing fiscal space in default, X_d, and its sign coincides with that of dX_d/dm. Two opposing forces operate here. First, stronger protection increases the entitlements guaranteed in default and directly squeezes fiscal space in default (corresponding to the direct effect in (25)). Second, stronger protection moves equilibrium entitlement setting itself (corresponding to the entitlement response in the same equation). In the small-α region assumed by Propositions 2 and 3, y^{E*} is decreasing in m (Section 4.6), so the second force works to relax fiscal space. The two forces point in opposite directions, and the sign of this channel is in general indeterminate. The change in fiscal space affects the successor generation's welfare through changes in its private consumption and public goods consumption.

The second term on the right-hand side is the effect of stronger protection changing the default probability. The coefficient $\ln(X_r/X_d)$ is positive and represents the size of the resources lost when default occurs. A rise in the default probability increases the frequency of the resource-poor state and lowers the successor generation's welfare. This channel therefore has the opposite sign to $d\delta^*/dm$: it is positive if stronger protection lowers the default probability and negative if it raises it.

As seen above, a change in the degree of protection m operates on both generations through multiple channels, and the sign of the combined effect is in general indeterminate. The two propositions below show that near the borrowing shutdown level ($m = \underline{m}$) and near full protection ($m = 1$), the signs of the welfare effects on the two generations coincide, and that the direction reverses between the two endpoints: near the lower end, strengthening protection, and near full protection, relaxing it, are each Pareto improving.

6.3 How stronger protection achieves a Pareto improvement near the borrowing shutdown level

As in the previous section, consider first the lower end of the analysis range, the neighborhood of the borrowing shutdown level $m = \underline{m}$. The following proposition holds.

Proposition 4. *Suppose the same assumptions and parameter conditions as in Proposition 2 hold: Assumption 1 is satisfied, z_1 is small enough that the borrowing shutdown level $\underline{m} \in (0, 1)$ is defined, and conditions (CP2a) and (CP2b) are satisfied. Then, if α is sufficiently small, in a neighborhood of the borrowing shutdown level $\underline{m}$,*

$$\frac{dV_I^*(m)}{dm} > 0 \qquad \textit{and} \qquad \frac{dV_S^*(m)}{dm} > 0$$

hold simultaneously. That is, a marginal strengthening of entitlement protection near $m = \underline{m}$ improves the welfare of both the issuing generation and the successor generation, achieving a Pareto improvement.

See Appendix E for the proof.

The conditions of this proposition are identical to those of Proposition 2. Conditions (CP2a) and (CP2b) guarantee that stronger protection lowers the equilibrium default probability in this region—$d\delta^*/dm < 0$. The proposition asserts that the same strengthening of protection also improves the welfare of both generations—that the fall in the default probability comes at the expense of neither.

The change in the issuing generation's welfare V_I

The three channels in (28) operate on the issuing generation's welfare. At the borrowing shutdown level, however, $d^* = 0$, and debt issuance is small in its neighborhood, so the bond price channel is weak: even if stronger protection lowers the default probability and raises the bond price, little additional revenue accrues when issuance itself is small. The change in welfare is therefore decided by the contest between the remaining two channels—the default-state consumption channel and the default-state public goods channel.

The former effect is positive. When the degree of protection m is small, the issuing generation's old-age consumption swings widely between y^{E*} and my^{E*} depending on the realization of z_2, and this uncertainty is a burden. Stronger protection narrows this swing and, at the same time, reduces the frequency of default. Moreover, in this region, where large entitlements have been promised in anticipation of weak protection, the power of stronger protection to reduce the default frequency is also strong: a marginal strengthening of protection reduces the fiscal outlay the period-2 government can save by defaulting by the full amount y^{E*}. This is the direct effect in (25).

The latter effect is negative: the more entitlements are paid, the less period-2 public goods are provided in default. Under the stated conditions the weight on public goods is small because α is small, but in this region the resources available in default are very scarce and the marginal utility of public goods consumption is high, so this cost remains non-negligible.

What decides between the two is the resources available in period 1, z_1. In an economy with small z_1, period-1 resources are precious and the borrowing incentive strong, so positive debt issuance occurs even when protection is weak and large entitlements have been set in anticipation of that weakness—that is, the borrowing shutdown level $\underline{m}$ is low. In that case the two sources of the positive effect seen above—narrowing the swing in old-age consumption and reducing the frequency of default—operate powerfully. The condition in Proposition 4 that "z_1 is sufficiently small" requires exactly this. Under this condition, the positive effect dominates the negative one, and a rise in m improves the issuing generation's welfare.

The change in the successor generation's welfare V_S

For the successor generation, stronger protection means two things: a change in fiscal space in default, $X_d(y^{E*})$, and a change in how likely default is to occur. The two channels in (30) correspond to these two.

The former barely operates. Stronger protection raises the fraction of entitlements paid in default—but the issuing generation does not take this as a given. Fiscal space in default is scarce to begin with, since income falls to the lower bound $\underline{z}$ and entitlements must still be paid, and if the total payment my^{E*} grows, public goods provision shrinks accordingly. The scarcer fiscal space is, the higher the marginal utility of public goods consumption, so this loss is large even for the issuing generation that set the entitlements. The issuing generation therefore scales down its entitlement setting by as much as protection strengthens (Section 4.6). The rise in the guaranteed fraction is absorbed by the contraction in the amount set, so the total payment in default, my^{E*}, and hence the fiscal space left to the successor generation, X_d, barely move.

The latter, by contrast, operates powerfully. The successor generation is the party that bears the consequences of default. If default occurs, income falls to the lower bound and the guaranteed entitlements must still be paid, leaving them with extremely little private consumption and few public goods. The scarcer resources are, the more each unit is worth, so the loss that a default inflicts on them is large—and so is the gain from its frequency falling.

In short, stronger protection leaves the fiscal space X_d remaining to the successor generation almost unchanged and moves only the probability of falling into that resource-poor state. This is why their ex ante welfare is determined almost entirely by whether default occurs, moving inversely with the default probability. Near the borrowing shutdown level, stronger protection lowers the default probability (Proposition 2), which directly benefits them.

Read together with Proposition 2, this proposition shows that near the borrowing shutdown level, stronger protection lowers the equilibrium default probability while improving the welfare of both generations. The fall in the default probability here comes at the expense of neither generation.

6.4 How weaker protection achieves a Pareto improvement near full protection

Consider next the upper end of the analysis range, the neighborhood of full protection, $m = 1$. The following proposition holds.

Proposition 5. *Suppose the same assumptions and parameter conditions as in Proposition 3 hold: Assumption 1 is satisfied, z_1 is small enough that the borrowing shutdown level $\underline{m} \in (0, 1)$ is defined, and condition (CP3) is satisfied. Then, if α is sufficiently small, in a neighborhood of full protection $m = 1$,*

$$\frac{dV_I^*(m)}{dm} < 0 \qquad \text{and} \qquad \frac{dV_S^*(m)}{dm} < 0$$

hold simultaneously. That is, a marginal relaxation of entitlement protection near $m = 1$ improves the welfare of both the issuing generation and the successor generation, achieving a Pareto improvement.

See Appendix F for the proof.

The conditions of this proposition are identical to those of Proposition 3. Condition (CP3) guarantees that relaxing protection lowers the equilibrium default probability in this region—$d\delta^*/dm > 0$. The proposition asserts that the same relaxation improves the welfare of both generations, that is, that the fall in the default probability at the upper end likewise comes at the expense of neither generation.

The change in the issuing generation's welfare V_I

As at the lower end, the three channels in (28) operate on the issuing generation's welfare, but in the region close to full protection each channel operates differently. Consider how the issuing generation's welfare changes through each channel when entitlement protection is relaxed—that is, m falls—in the region close to full protection.

Look first at the default-state consumption channel. This channel consists of the effect of a lower m reducing the level of entitlements guaranteed in default and the effect of raising the frequency of default (the direct effect in (25)), both of which work to lower the issuing generation's welfare. Near full protection, however, the latter barely operates: even if default occurs, its own entitlements are almost fully protected, so the loss from one extra default occurring has vanished. The effect of relaxing entitlement protection through this channel is therefore limited to the negative effect of a lower guaranteed level of entitlements in default.

Next, the default-state public goods channel. Relaxation reduces the entitlements paid in default and increases public goods provision by that amount, so relaxing protection has a positive effect—and the benefit is large. Near full protection, almost all of the scarce fiscal space in default is absorbed by entitlement payments to the issuing generation itself, and little is left to finance public goods. With its own private consumption already amply secured while public goods are exhausted, public goods are more valuable at the margin.

What remains is the bond price channel. Relaxing protection enlarges the fiscal outlay the period-2 government can save by defaulting by y^{E*}, directly raising the default probability (the direct effect in (25)). The bond price therefore falls, and by making it less favorable to raise resources in period 1, this exerts a negative effect on the issuing generation's welfare.[11] This channel was weak at the lower end, but here it is non-negligible: near full protection, equilibrium debt issuance is large, so the price decline bites strongly through the amount raised.

Whether relaxing entitlement protection raises or lowers the issuing generation's welfare thus depends on the balance between the negative effects through the default-state consumption channel and the bond price channel and the positive effect through the default-state public goods channel. As in the case near the borrowing shutdown level, what decides the balance is the resources available in period 1, z_1. In an economy with small z_1, period-1 resources are precious and the loss from a lower bond price is large, so the negative effects dominate and relaxation actually worsens welfare. Condition (CP3), $z_1 > (5\underline{z} - \overline{z})/4$, requires that z_1 not be that small. When it holds, the positive effect dominates, and relaxing protection improves the issuing generation's welfare.

[11]Note that what operates here is the direct effect of m, not the change in the equilibrium default probability $\delta^*(m)$. Because the issuing generation optimizes over $\{d, y^E\}$, the envelope theorem implies that the effects operating through changes in the equilibrium policies cancel out.

The change in the successor generation's welfare V_S

The configuration seen in the previous subsection—fiscal space in default X_d barely moving while only the probability of default moves—is not special to the lower end. Near full protection as well, if the guaranteed fraction falls, the issuing generation raises its entitlement setting, so the total paid in default, my^{E*}, barely changes, and neither does the scarcity of the resources that remain. What relaxation moves is the likelihood of default, which it lowers in this region (Proposition 3). Relaxing protection therefore improves the successor generation's welfare.

Read together with Proposition 3, this proposition shows that near full protection, relaxing protection lowers the equilibrium default probability while improving the welfare of both generations. The fall in the default probability at the upper end, too, comes at the expense of neither generation.

6.5 Why the two generations' interests coincide at both ends

Taken together, Propositions 4 and 5 imply that both $V_I^*(m)$ and $V_S^*(m)$ slope upward near the borrowing shutdown level and downward near full protection. That is, where entitlement protection is weak, strengthening it achieves a Pareto improvement, and where it is strong, relaxing it does. For the welfare of either generation, entitlement protection should be neither too strong nor too weak. This subsection lays out why the interests of the two generations coincide at both ends.

For the successor generation, the answer is simple. They are the party bearing the consequences of default, and the fiscal space X_d remaining in default barely moves with the degree of protection: even if the guaranteed fraction rises, the issuing generation scales down its entitlement setting by that much, so the total paid in default is unchanged. What the degree of protection moves is the likelihood of default, and their welfare is determined inversely to it. Since the default probability rises at both ends, as seen in the previous section, their welfare is low at both ends.

For the issuing generation, the answer is not so simple, because strengthening protection carries both a benefit and a cost for them. The benefit is that their own entitlements escape being cut when default occurs. In the region of weak protection, their old-age consumption differs greatly depending on whether default occurs, and this uncertainty is a heavy burden. Stronger protection, moreover, also reduces the frequency of default itself. In this region the benefit is large, and stronger protection raises their own welfare.

This benefit, however, saturates as protection strengthens. Near full protection, entitlements are already almost fully protected, and little remains to be gained from protecting them further. The cost, meanwhile, persists: the period-2 resources in default are scarce—income has fallen to its lower bound and strongly guaranteed entitlements must still be paid—so public goods provision is small. With their own private consumption already amply secured while public goods are exhausted, public goods are more valuable to the issuing generation at the margin. Relaxing protection and redirecting resources to public goods is preferable even for the very party that set the entitlements.

Protection that is too weak exposes their own old-age consumption to risk; protection that is too strong sacrifices public goods in default. Both raise the frequency of default, and both lower the welfare of each generation. This is why the degrees of protection that maximize each generation's welfare both lie at interior points, and why the two generations' interests coincide near the two endpoints. All this answers the question posed at the start of this section—is a lower default probability really desirable? Adjustments of the degree of entitlement protection that lower the default probability near either endpoint constitute an efficiency improvement that sacrifices neither generation. The policy implications in light of real-world institutions are discussed in Section 8.

6.6 A numerical illustration of Propositions 4 and 5

Under the same baseline as the numerical illustration of Propositions 2 and 3 in Section 5.5, $\underline{z} = 1,\ \overline{z} = 6.5,\ z_1 = 2.5,\ \alpha = 0.03$, Propositions 4 and 5 are now confirmed numerically. Figure 6 plots the issuing generation's welfare $V_I^*(m)$ (panel (a)) and the successor generation's welfare $V_S^*(m)$ (panel (b)), computed numerically from the full equilibrium.

Both welfare functions are hump-shaped with interior peaks. Near the borrowing shutdown level both slope upward, so strengthening protection improves the welfare of both generations simultaneously (Proposition 4); near full protection both slope downward, so relaxing protection improves the welfare of both generations simultaneously (Proposition 5). Because the two peaks do not coincide, in a narrow central region the generations' interests diverge and adjusting the degree of protection is not Pareto improving. The claims of both propositions are thus confirmed numerically.

7 The role of the parameter conditions and robustness

This section uses numerical examples to examine what the parameter conditions underlying the main results actually require. Propositions 2 and 3 showed that when z_1 and α are sufficiently small and conditions (CP2a), (CP2b), and (CP3) hold, $\delta^*(m)$ slopes downward near the borrowing shutdown level $\underline{m}$ and upward near full protection $m = 1$. Below, starting from the baseline $\underline{z} = 1,\ \overline{z} = 6.5,\ z_1 = 2.5,\ \alpha = 0.03$ and moving only z_1 or only α in the direction that violates these conditions, we show that the shape of $\delta^*(m)$ indeed changes. The conditions on the sizes of z_1 and α are thus not formal

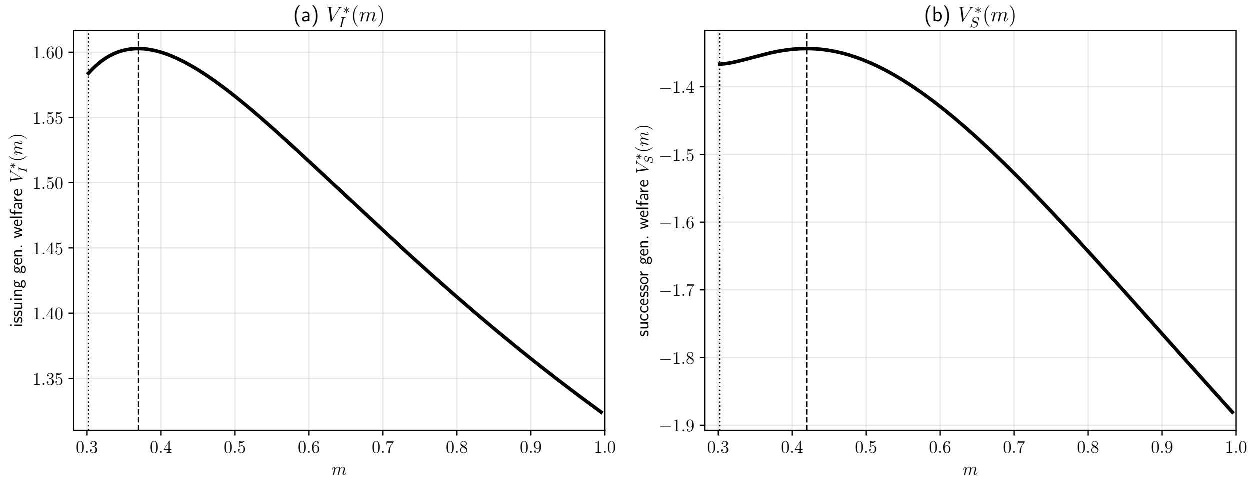

Figure 6: A numerical illustration of Propositions 4 and 5 (baseline: $\underline{z} = 1$, $\overline{z} = 6.5$, $z_1 = 2.5$, $\alpha = 0.03$; $p_z \approx 0.182$). The vertical dotted line marks the borrowing shutdown level $\underline{m} \simeq 0.30$, and the vertical dashed lines the peak of each welfare function. Panel (a): the issuing generation's welfare $V_I^*(m)$. Panel (b): the successor generation's welfare $V_S^*(m)$. Both welfare measures omit constant terms.

assumptions made merely to derive the results: they have substantive content, in the sense that when they fail, the very shape of $\delta^*(m)$ changes.

When the conditions fail: two deviating cases for $\delta^*(m)$

Figure 7 places the baseline U-shape (panel (a)) alongside two cases in which α or z_1 is moved in the condition-violating direction (panels (b) and (c)). For these two cases, figures plotting not only $\delta^*(m)$ but all the equilibrium quantities $(d^*, y^{E*}, \delta^*, V_I^*, V_S^*)$ as functions of m are placed in Appendix G (Figures 10 and 11).

First, raising only α from the baseline ($\alpha : 0.03 \to 0.6$) eliminates the upward slope at the full-protection end, and $\delta^*(m)$ becomes downward sloping. What pushed δ^* up near full protection was the borrowing response: the stronger the protection, the smaller the loss the issuing generation suffers when default occurs, and hence the larger debt issuance. In an economy with a strong preference for public goods, however, the price of raising the default probability by issuing more debt is high: if default occurs, period-2 fiscal space is scarce and almost no public goods can be provided. In this economy, debt issuance itself stays small and rises little as protection strengthens. What remains is only the restraining side—stronger protection reduces the entitlements flowing into the general budget in default, $(1 - m)y^{E*}$, weakening the period-2 government's default incentive—so the default probability falls consistently in the degree of entitlement protection.

Second, raising only z_1 from the baseline ($z_1 : 2.5 \to 5.0$) instead eliminates the downward slope at the borrowing shutdown end, and $\delta^*(m)$ becomes upward sloping. In an economy rich in period-1 resources, the need to borrow is small, so the borrowing shutdown level at which issuance reaches zero itself shifts up ($\underline{m}$: 0.30 $\to$ 0.49). At this end, protection is already fairly strong and entitlements are set small. The restraint from strengthening protection further and squeezing entitlements—the direct effect and the entitlement response—is therefore weak to begin with. The borrowing response—stronger protection lowering the default probability, raising the bond price, and encouraging issuance—is, on the other hand, almost as large as in the baseline. The borrowing response thus dominates the direct effect and the entitlement response, and stronger protection raises the default probability.

The shape of $\delta^*(m)$ is thus determined by two factors: the strength of the preference for public goods α and the resources available in period 1, z_1. In economies with a strong public goods preference, stronger protection consistently lowers the default probability; in economies rich in period-1 resources, it consistently raises it. The main results of this paper concern economies that are neither—with a weak public goods preference and scarce period-1 resources—where the relationship between the degree of protection and the default probability is U-shaped and the risk-minimizing degree of protection lies at an interior point.

The parameter range over which the results hold

How far do the main results extend when z_1 and α move together? Figure 8 numerically delineates, in the (z_1, α) plane, the region where an interior equilibrium exists at every point of $m \in [\underline{m}, 1]$ and all the claims of Propositions 2–5 hold. The conditions and the procedure used for the determination are given in Appendix G.

The upper bound on α is highest around $z_1 \simeq 2.85$, reaching 0.149, so the claims survive up to nearly five times the baseline level (fixing z_1 near this value and moving α up to 0.5, the claims never hold in the region above 0.15). The two deviating cases treated in this section both lie outside this region.

This numerical confirmation carries more significance than the interpretation of the conditions. The proofs of Propositions 2–5 use the $\alpha \to 0$ limit: because the equilibrium is defined implicitly as a point jointly satisfying two first-order conditions, pinning down the signs of the comparative statics with $\alpha > 0$ is difficult, and only in the limit does a closed-form determination become possible. But the conclusions are not confined to that limit. As Figure 8 shows, even with α fixed at levels away from zero, the signs of δ^* and of both generations' welfare at the two endpoints are preserved.

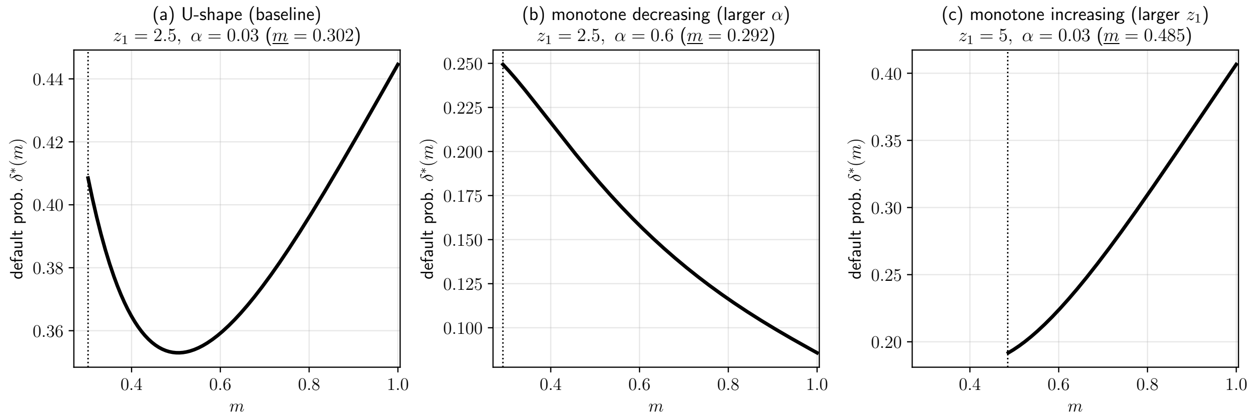

Figure 7: The equilibrium default probability $\delta^*(m)$ when only z_1 or only α is moved from the baseline ($\underline{z} = 1,\ \overline{z} = 6.5$ are common; the horizontal axis range is the same across the three panels). The vertical dotted line marks each panel's borrowing shutdown level $\underline{m}$. Panel (a): baseline ($z_1 = 2.5,\ \alpha = 0.03$), U-shaped (reproduced from panel (a) of Figure 5). Panel (b): only α raised ($\alpha = 0.6$), monotonically decreasing. Panel (c): only z_1 raised ($z_1 = 5.0$), monotonically increasing. See Figures 10 and 11 for the details of the equilibrium quantities in each case.

Interpreting the parameter levels and situating the model

The model in this paper is simplified so that the mechanism can be examined analytically, and it is not structured so that its parameters can be mapped to real-world data. Indeed, under log utility the period-2 government's optimization implies the identity $\alpha = g_2^*/c_2^{y*}$, so α corresponds to an observable quantity, the ratio of public goods provision to young-generation consumption. The levels used in this paper are smaller than the values of this ratio measured for actual emerging market economies.

That said, Bouton et al. (2020), on which this paper builds, derive their mechanism analytically without specifying the utility function and conduct no quantitative analysis. By specifying log utility, this paper has acquired a comparable quantity; but models of this kind aim to isolate mechanisms, and this paper follows that practice. There are properties that must be understood before proceeding to quantitative analysis, and it is these that this paper has clarified.

8 The degree of entitlement protection m: real-world determinants and policy implications

In the model, m is the institutional parameter representing the fraction of entitlements that is honored even when the government falls into default (or severe fiscal distress). In reality, this "resilience of entitlements under fiscal stress" is determined, first, by the legal status with which benefit rights are designed—whether they are protected by the constitution, protected as property rights, or treated as

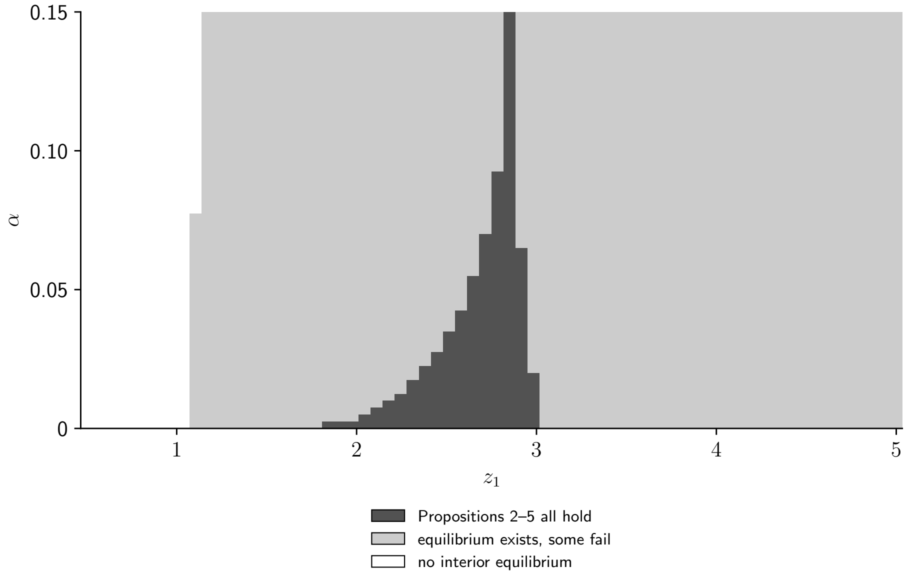

Figure 8: The range of (z_1, α) over which all the claims of Propositions 2–5 hold ($\underline{z} = 1$, $\overline{z} = 6.5$). Dark gray marks points where all four claims hold, light gray points where an interior equilibrium exists but some claims fail, and white points where an interior equilibrium fails to obtain for part of $[\underline{m}, 1]$.

mere gratuities—and, second, by how far the judiciary actually upholds that protection in times of fiscal crisis. Neither can be adjusted by the government in the short run in response to current fiscal conditions. At the same time, neither is an immutable given: through constitutional or statutory amendment, or the accumulation of judicial decisions, m can be redesigned in the long run. It is because m is an object of institutional design in this sense that this paper conducts comparative statics with respect to it.

The level of m, moreover, differs greatly across countries and states. This section draws on two sets of evidence—a comparison across U.S. states and a comparison across European countries during the euro crisis—to organize where m is large (entitlements strongly protected) and where it is small.

8.1 The determinants of m and their international diversity

The first determinant: the legal status of benefit rights

Monahan (2010), drawing on U.S. state public employee pensions, organizes the degree of protection of public pensions—one form of entitlement—into three categories. First, the strongest protection is protection by the state constitution. The constitutions of New York and Illinois explicitly provide that pension benefits "shall not be diminished or impaired." Under this regime, not only benefits already accrued but even the benefit rules governing future accruals are protected, making ex post cuts extremely difficult. This corresponds to states with a large m.

Second, protection is intermediate where benefit rights are constructed as property interests, as in Connecticut and Maine. Here rights are protected against improper taking, but if the government demonstrates a reasonable basis such as fiscal consolidation, adjustments such as raising the retirement age through ordinary legislation are permitted. This corresponds to states with an intermediate m.

Third, protection is weakest where benefits are treated as gratuities from the state government, as in some pension plans in Texas and Indiana. Here no legal right arises at all until actual retirement, and the government can cut or abolish benefits at any time by simple legislation. This corresponds to states with a small m.

While this typology is based on U.S. states, the distinction it employs—whether benefit rights receive constitutional protection, are constructed as property rights, or remain benefits that can be revised or repealed by ordinary legislation—is not specific to U.S. federalism. Across countries, too, what protects benefit rights differs. Brazil falls into the first category (strongest protection): the 1988 Constitution regulates pension rights in detail, and changes in benefit rules have often required constitutional amendment (Matijascic and Kay 2008). Germany falls into the second (intermediate protection): accrued pension expectancies are covered by the Basic Law's property guarantee, but this protection is not absolute, and no particular benefit level is constitutionally required (Federal Constitutional Court of Germany 1999). The United Kingdom falls into the third (weakest protection): the basic state pension is set by ordinary statute, and the eligibility age and uprating rules have been changed by legislation (House of Commons Library 2026). Reflecting these differences in how rights are stipulated, the degree of protection m varies widely across countries as well.

The second determinant: the judiciary's stance in times of crisis

Legal status, however, defines only the protection on paper. Whether it is actually upheld in a fiscal crisis depends on the stance the judiciary takes toward cutback measures. After the euro crisis, several European countries implemented public pension cuts as part of fiscal consolidation, and judicial attitudes toward these measures differed markedly across countries.

As Poulou (2014) discusses, Portugal's Constitutional Court exercised relatively active judicial review of austerity measures, striking down certain cuts to public sector wages and pensions as unconstitutional on grounds such as the principle of the protection of legitimate expectations. Greece's Council of State, by contrast, took a judicially deferential stance in the early phase of the crisis, upholding most austerity measures including pension cuts. As a result, large-scale pension cuts in Greece were legally sanctioned and carried out. Although the court later found some wage and pension cuts unconstitutional, the judicial responses of the two countries in the early phase of the crisis differed conspicuously.

That is, even facing similar pressure to cut, how much is actually protected ex post differs with the stance of the judiciary. Interpreted through the model, early-crisis Portugal corresponds to a situation with a relatively large m, and Greece to one with a relatively small m.

8.2 Policy implications

In sum, m differs greatly with a country's institutions: it is high where benefit rights are constitutionally protected and the judiciary upholds that protection even in crisis, and low where benefit rights are treated as mere gratuities or the judiciary defers under crisis conditions. Placed against this institutional diversity, the main result of this paper—that the risk-minimizing m lies in the middle (Propositions 2 and 3)—carries a clear policy implication: from the standpoint of default risk, neither too strong a protection of entitlements (an excessively high m) nor too weak a protection (an excessively low m) is optimal. Moreover, the real-world policy levers that move m—how benefit rights are legally defined, how far they are protected by the constitution, and how judicial review in times of crisis is designed—correspond directly to the parameter m targeted by the comparative statics of this paper.

What can be read from the model, however, is not the desirable level of protection but the direction of adjustment. Where the risk-minimizing degree of protection lies depends not only on the degree of protection itself but also on how much room the government has left to borrow, how heavily public goods are valued, and how wide income fluctuations are (Section 7). One therefore cannot mechanically assert that "countries with strong protection should relax it." What this paper provides is a criterion for telling which margin is operative. In an economy where protection is weak and enormous promises stand to be lost in default, strengthening protection may lower default risk. Conversely, in an economy where protection is nearly complete and entitlements no longer act as a check on borrowing, relaxing protection may lower it.

Still more important for policy is that these adjustments are not zero-sum. Entitlement reform is often framed as a conflict between current beneficiaries and future generations. Propositions 4 and 5, however, show that near the two ends, adjusting the degree of protection in the direction that lowers the default probability raises the welfare of both generations simultaneously. The distinctive significance of this result is that it speaks not only to the desirability of reform but also to its political feasibility.

That said, the model is a simple two-period framework without private saving and does not deliver a numerical value for the optimal degree of protection. The implications stated here should be read as qualitative guidance on which direction of adjustment benefits both generations.

9 Conclusion

This paper introduced the possibility of endogenous sovereign default into a political economy model in which both government debt and entitlements are endogenous, and analyzed the effect of the degree of entitlement protection m—the fraction of entitlements guaranteed even in default—on the equilibrium default probability and intergenerational welfare. Whereas existing political economy models have treated only the special case $m = 1$, in which entitlements are always honored, this paper extended the analysis to $m < 1$ and positioned the degree of protection itself as the parameter of the comparative statics.

The first result is that the relationship between the degree of protection m and the equilibrium default probability $\delta^*(m)$ is non-monotonic, with the sign reversing at the two ends (Propositions 2 and 3): near the borrowing shutdown level $\underline{m}$, strengthening protection lowers the equilibrium default probability, and near full protection $m = 1$, relaxing it does.

The reversal arises because the degree of protection shifts the composition of the instruments through which the issuing generation extracts resources from the future. Of the fiscal outlays saved by default, debt is written off in full while only the unprotected portion of entitlements is. Hence when protection is weak, extraction through entitlements is cheap and large entitlements are set; when protection is strong, entitlements are restrained, but the issuing generation, no longer fearing default, increases debt issuance instead. Protection that is too weak inflates entitlements and protection that is too strong inflates debt, and at either end the fiscal outlay saved by default becomes large. This is why the degree of protection that minimizes the default probability lies at neither endpoint but strictly between them.

The second result is that near the two endpoints, adjusting m in the direction that lowers the default probability raises the welfare of the issuing generation and the successor generation simultaneously—a Pareto improvement (Propositions 4 and 5). Because fiscal space in default barely moves with the degree of protection, the successor generation's ex ante welfare is dominated by the probability of default, and the first result translates directly into their welfare. For the issuing generation, too, the benefit of stronger protection—its own entitlements being protected even when default occurs—saturates as protection strengthens, while its cost—the scarce resources in default being absorbed by payments to itself, draining the funding for public goods—persists. The degrees of protection that maximize each generation's welfare therefore also lie at neither endpoint but strictly between them.

The message of the paper can thus be summarized in three points. First, from the standpoint of minimizing default risk, entitlement protection should be neither too strong nor too weak. Second, from the standpoint of welfare as well, entitlement protection should be neither too strong nor too weak. Third, and most importantly, institutional reform of entitlement protection can achieve a Pareto improvement that benefits both the current and the future generation. In intergenerational political economy, the choice of the degree of entitlement protection is often framed as a zero-sum redistribution problem; the results of this paper show that this framing is not necessarily correct.

Finally, the limitations of this paper and directions for future research are as follows. First, the paper treated the degree of protection m as an exogenous institutional parameter. As seen in Section 8, the real-world m is shaped by policy variables such as the strength of constitutional and statutory protection, indexation rules, and the design of pension reform, and endogenizing these choices themselves as a political process is a natural extension. Second, the paper assumed that households can neither save nor borrow privately—the case of extreme capital market imperfection, the limit of the setting in Bouton et al. (2020). In their model, however, the strength of the imperfection itself played an important role in characterizing the equilibrium. Extending the analysis to an economy where households can save and borrow on terms less favorable than government bonds, and examining how the degree of imperfection interacts with the effects of the degree of protection, is a task worth undertaking. Third, the paper uses a two-period model; an extension to an infinite-horizon model would make it possible to ask whether similar results obtain in an environment where debt can be rolled over.

Fourth, Propositions 2 and 3 deliver only the signs near the two endpoints, and the behavior of $\delta^*(m)$ in between is outside the scope of this paper's analysis. Fifth, the paper concentrates on extracting the mechanism analytically and conducts no quantitative analysis. As in much of the sovereign default literature, building a richer model suited to calibration and analyzing it in an environment consistent with real-world data is called for. Quantitatively identifying the level of protection that minimizes default risk is a task for that next step.

Appendix

Throughout the Appendix, since the domains of the parameters are fixed, as in $\alpha > 0$ and $m \in [\underline{m}, 1]$, all limits are taken from within these domains, and subscripts indicating the direction of a limit are omitted. Derivatives with respect to m at the interval endpoint $m = \underline{m}$ denote right derivatives. Whenever the order of limits matters, it is stated explicitly.

A Derivation of the period-1 first-order conditions

We derive (18), (19), and (20) from problem (17). First, write the objective as an explicit function of $\{d, g_1, y^E, c_1^o\}$. The public goods term in default is

$$\ln g_{2,d}^*(y^E) = \ln A + \ln X_d(y^E).$$

Meanwhile, noting that $\int \ln u \, du = u \ln u - u$, the conditional expectation in the non-default case is computed as

$$\begin{aligned}
&E_{z_2}\left[\ln g_{2,r}^*(z_2, d, y^E) \mid z_2 \geq \hat{z}(d, y^E)\right] \\
&= \ln A + \frac{1}{\overline{z} - \hat{z}(d, y^E)} \int_{\hat{z}(d,y^E)}^{\overline{z}} \ln X_r(z_2, d, y^E) dz_2 \\
&= \ln A + \frac{1}{\overline{z} - \hat{z}(d, y^E)} \cdot \Big[X_r(\overline{z}, d, y^E) \ln X_r(\overline{z}, d, y^E) - X_r(\overline{z}, d, y^E) \\
&\qquad\qquad - X_r(\hat{z}(d, y^E), d, y^E) \ln X_r(\hat{z}(d, y^E), d, y^E) + X_r(\hat{z}(d, y^E), d, y^E)\Big].
\end{aligned}$$

In substituting this into (17), two relations are used. First, from $\delta(d, y^E) = p_z \left[\hat{z}(d, y^E) - \underline{z}\right]$ and $p_z = 1/(\overline{z} - \underline{z})$,

$$1 - \delta(d, y^E) = p_z \left[\overline{z} - \hat{z}(d, y^E)\right]$$

holds. Second, substituting (13),

$$X_r\big(\hat{z}(d, y^E), d, y^E\big) = \hat{z}(d, y^E) - d - y^E = \underline{z} - my^E = X_d(y^E)$$

holds. The second relation shows that fiscal space in the non-default case, evaluated at the default threshold, coincides with fiscal space in default; this is just a restatement of the fact that the threshold is determined by the default condition $V_S^d(y^E) = V_S^r(z_2, d, y^E)$. Using these relations and collecting constants, the objective can be rewritten as

$$
\begin{aligned}
V_I = \;& \ln\bigl(z_1 + q(d, y^E)d - c_1^o - g_1\bigr) + \alpha \ln g_1 + \alpha \ln A \\
& + \delta(d, y^E) \ln m + \ln y^E \\
& + \alpha\delta(d, y^E) \ln X_d(y^E) \\
& + \alpha p_z \cdot \Bigl[X_r(\overline{z}, d, y^E) \ln X_r(\overline{z}, d, y^E) - X_r(\overline{z}, d, y^E) - X_d(y^E) \ln X_d(y^E) + X_d(y^E)\Bigr]. \qquad \text{(A1)}
\end{aligned}
$$

Differentiating this objective with respect to g_1 yields $1/c_1^y = \alpha/g_1$. Combining this with $c_1^o = 0$ and the period-1 resource constraint $c_1^y = z_1 + q(d, y^E)d - g_1$ gives (18). Next, differentiating with respect to d and y^E, substituting (18) and $1/c_1^y = (1+\alpha)/\left[z_1 + q(d, y^E)d\right]$, and rearranging yields (19) and (20).

B Proof of Proposition 1

The proof of Proposition 1 proceeds in three steps, Step 1 through Step 3. Each step establishes one lemma—Lemma 1, Lemma 2, and Lemma 3, respectively—and combining them at the end yields the claim.

Step 1: Existence and uniqueness of $y^E(d)$

We first establish the following lemma.

Lemma 1. *Suppose Assumption 1 holds and z_1 is sufficiently small. Then there exists $d_c(m) \in (0, \overline{z} - \underline{z}]$ such that for each $d \in [0, d_c(m))$ there is a unique $y^E(d) \in \Omega$ satisfying $G(d, y^E) = 0$, and $\delta(d, y^E(d)) \to 1$ as $d \to d_c(m)$. We call $d_c(m)$ the* certain-default boundary.

Proof of Lemma 1. Fix d satisfying $0 \le d < d_h(0) = 1/p_z = \overline{z} - \underline{z}$. This upper bound delimits the range of d for which a feasible y^E exists given the fixed d: from (22), if $d \ge \overline{z} - \underline{z}$, then $\delta < 1$ fails for every $y^E \ge 0$ and no feasible equilibrium obtains.

Step 1-A. Boundary behavior.

As $y^E \downarrow 0$, note that

$$
\lim_{y^E \downarrow 0} \delta(d, y^E) = p_z d, \quad \lim_{y^E \downarrow 0} q(d, y^E) = 1 - p_z d, \quad \lim_{y^E \downarrow 0} X_d(y^E) = \underline{z}, \quad \lim_{y^E \downarrow 0} X_r(\overline{z}, d, y^E) = \overline{z} - d.
$$

From these relations, (#$G3$) in (20) diverges to ∞ while all the remaining terms converge to finite values. Hence

$$\lim_{y^E \downarrow 0} G(d, y^E) = +\infty.$$

Next, consider the upper boundary. For fixed d, the upper end of the feasible range of y^E is the smaller of the bounds implied by conditions (22) and (23). Writing the former, solved for y^E, as $y^E_{ub}(d) = (\overline{z} - \underline{z} - d)/(1 - m)$, this is exactly the value of y^E at which $\delta(d, y^E) = 1$. The upper end is thus $\min\{y^E_h,\ y^E_{ub}(d)\}$, where either $\delta = 1$ or $X_d = 0$ holds. Define the d at which the two boundaries coincide, that is, the d satisfying $y^E_h = y^E_{ub}(d)$, as

$$d_f \equiv d_h(y^E_h) = \overline{z} - \frac{\underline{z}}{m}. \tag{B1}$$

With this d_f, the upper end is y^E_h for $d < d_f$ and $y^E_{ub}(d)$ for $d > d_f$.

Consider first $d \leq d_f$, the case in which raising y^E reaches the upper end y^E_h (where $X_d \to 0$) first. At the upper boundary of y^E, δ, q, and $X_r(\overline{z}, d, y^E)$ all converge to finite values. Hence (#$G4$) and (#$G5$) in (20) diverge to $-\infty$ while the remaining terms converge, so $G(d, y^E) \to -\infty$.

Consider next $d > d_f$, the case in which raising y^E reaches the upper end $y^E_{ub}(d)$ (where $\delta = 1$, $q = 0$) first. Since d is fixed in the range $d < \overline{z} - \underline{z}$, this upper end is positive. Moreover, this upper end is the y^E that yields $\delta = 1$ ($q = 0$), and by (B3) we have $X_r = X_d$ there, so G at the upper end is

$$G(d, y^E_{ub}(d)) = \underbrace{-\frac{(1+\alpha)(1-m)p_z d}{z_1}}_{(\#G1)} + \underbrace{(1-m)p_z \ln m}_{(\#G2)} + \underbrace{\frac{1}{y^E_{ub}(d)}}_{(\#G3)} - \underbrace{\frac{\alpha m}{X_d(y^E_{ub}(d))}}_{(\#G4)}.$$

Viewing this value at the upper end as a function of $d \in (d_f, \overline{z} - \underline{z})$, examine its behavior at the two ends. As $d \downarrow d_f$, the upper end $y^E_{ub}(d)$ approaches $y^E_h = \underline{z}/m$ and $X_d(y^E_{ub}(d)) = \underline{z} - m y^E_{ub}(d) \to 0$, so (#$G4$) $= -\alpha m / X_d \to -\infty$; the other terms converge, so $G(d, y^E_{ub}(d)) \to -\infty$. As $d \uparrow \overline{z} - \underline{z}$, the upper end $y^E_{ub}(d) \to 0$, so (#$G3$) $= 1/y^E_{ub} \to +\infty$ while the other terms converge, so $G(d, y^E_{ub}(d)) \to +\infty$.

Since $G(d, y^E_{ub}(d))$ is continuous in $d \in (d_f, \overline{z} - \underline{z})$, tending to $-\infty$ as $d \downarrow d_f$ and to $+\infty$ as $d \uparrow \overline{z} - \underline{z}$, the intermediate value theorem yields a d at which $G(d, y^E_{ub}(d)) = 0$. Define the smallest such d as $d_c(m) \in (d_f, \overline{z} - \underline{z})$. This is the certain-default boundary of Lemma 1 (that $\delta \to 1$ as $d \to d_c(m)$ is verified in Step 1-C). Then $G(d, y^E_{ub}(d)) < 0$ for every $d \in (d_f, d_c(m))$. Since G at the upper end was $-\infty$ for $d \leq d_f$ as well (previous paragraph), G at the upper end of y^E is negative on all of

$d \in [0, d_c(m))$.

It follows that for every $d \in [0, d_c(m))$, $G(d, y^E) \to +\infty$ as $y^E \to 0$ and $G(d, y^E) < 0$ as y^E approaches the upper boundary. Hence for every $d \in [0, d_c(m))$ there exists at least one $y^E(d)$ within the feasible range satisfying $G(d, y^E) = 0$: whether the upper end is y^E_h (the case $d \le d_f$) or $y^E_{ub}(d)$ (the case $d \in (d_f, d_c(m))$), G crosses zero in the interior of the feasible range.

Figure 9(a) illustrates the analysis of Step 1-A numerically. Fixing $m = 0.5$ under the baseline parameters ($\underline{z} = 1,\ \overline{z} = 6.5,\ z_1 = 2.5,\ \alpha = 0.03$), we have $y^E_h = \underline{z}/m = 2$, $d_f = \overline{z} - \underline{z}/m = 4.5$, and $d_c(m) \simeq 4.551$. The figure depicts one instance of each of the two cases above. For $d = 0.5$ ($< d_f$) the upper end is $y^E_h = 2$, where $X_d \to 0$ and hence $G \to -\infty$. For $d = 4.53$ ($\in (d_f, d_c(m))$) the upper end is $y^E_{ub}(4.53) = 1.94$, where $X_d > 0$ is preserved and G stops at a finite negative value ($\simeq -0.21$) (open circle). In both cases $G \to +\infty$ as $y^E \downarrow 0$, so G crosses zero in the interior of the feasible range (filled circles, at $y^E(d) \simeq 1.98$ and 1.90, respectively). That $G(d, y^E)$ is strictly decreasing in y^E is shown in Step 1-B.

Step 1-B. Strict monotonicity in y^E.

Partially differentiating the right-hand side of (20) with respect to y^E yields

$$\begin{aligned}\frac{\partial G(d, y^E)}{\partial y^E} = & -\frac{(1+\alpha)(1-m)^2 p_z^2 d^2}{[z_1 + q(d, y^E)d]^2} - \frac{1}{(y^E)^2} - \alpha \cdot \delta \cdot \frac{m^2}{X_d(y^E)^2} \\ & + \alpha p_z \cdot \Big[\frac{1}{X_r(\overline{z}, d, y^E)} - \frac{m(2-m)}{X_d(y^E)}\Big]. \end{aligned} \tag{B2}$$

Monotonicity is shown in the limit $z_1 \to 0$. As $z_1 \to 0$, the first term converges to $-(1+\alpha)(1-m)^2 p_z^2/q^2$, so (B2) can be arranged as a first-degree polynomial in α,

$$\lim_{z_1 \to 0} \frac{\partial G}{\partial y^E} = \underbrace{-\frac{(1-m)^2 p_z^2}{q^2} - \frac{1}{(y^E)^2}}_{(\#1)} + \alpha \underbrace{\left[-\frac{(1-m)^2 p_z^2}{q^2} - \frac{\delta m^2}{X_d(y^E)^2} + p_z \Big(\underbrace{\frac{1}{X_r(\overline{z}, d, y^E)} - \frac{m(2-m)}{X_d(y^E)}}_{(\#3)}\Big)\right]}_{(\#2)}.$$

Since the part (#1) is negative, showing (#2) ≤ 0 establishes $\lim_{z_1 \to 0} \partial G/\partial y^E < 0$ for every $\alpha \ge 0$.

We split by the sign of (#3) in the final bracket of (#2). If (#3) ≤ 0, then (#2) is a sum of non-positive terms, so (#2) < 0 and $\lim_{z_1 \to 0}(\partial G/\partial y^E) < 0$.

Consider next the case (#3) > 0. From $X_r(\overline{z}, d, y^E) = \overline{z} - d - y^E$ and $X_d = \underline{z} - my^E$,

$$X_r - X_d = (\overline{z} - \underline{z}) - [d + (1-m)y^E] = \frac{1-\delta}{p_z} = \frac{q}{p_z}, \tag{B3}$$

holds. Using this,

$$(\#3) > 0 \;\Leftrightarrow\; X_r > \frac{q}{(1-m)^2 p_z}.$$

Substituting this into the third term of (#2) gives

$$\begin{aligned}
(\#2) &< -\frac{(1-m)^2 p_z^2}{q^2} - \frac{\delta m^2}{X_d^2} + \frac{(1-m)^2 p_z^2}{q} - \frac{p_z m(2-m)}{X_d} \\
&= -(1-q)\frac{(1-m)^2 p_z^2}{q^2} - \frac{\delta m^2}{X_d^2} - \frac{p_z m(2-m)}{X_d} \\
&< 0.
\end{aligned}$$

Hence $(\#2) < 0$ in either case, so $\lim_{z_1 \to 0} \partial G/\partial y^E < 0$, and by continuity, if z_1 is sufficiently small, $G(d, y^E)$ is strictly decreasing in y^E over the entire feasible range (Figure 9(a)).

In sum, when z_1 is sufficiently small, for fixed $d \in [0, d_c(m))$, $G(d, y^E)$ diverges to $+\infty$ at the lower boundary $y^E \downarrow 0$ (Step 1-A) and takes a negative value at the upper boundary, namely the largest y^E keeping $(d, y^E) \in \Omega$ for the given d. Moreover, $G(d, y^E)$ is continuous and strictly decreasing in y^E (Step 1-B). By the intermediate value theorem, therefore, a unique $y^E(d)$ satisfying $G(d, y^E(d)) = 0$ exists in the interior of the range of y^E keeping $(d, y^E) \in \Omega$ (see Figure 9(a)). This establishes the existence and uniqueness of $y^E(d) \in \Omega$ for each $d \in [0, d_c(m))$.

Step 1-C. $\delta \to 1$ **at the certain-default boundary.**

Finally, we show the remaining claims of Lemma 1: $\delta(d, y^E(d)) < 1$ for $d \in [0, d_c(m))$, and $\delta(d, y^E(d)) \to 1$ as $d \to d_c(m)$.

The key is the equivalence obtained from $\delta(d, y^E) = p_z[d + (1-m)y^E]$ and $1/p_z = \overline{z} - \underline{z}$,

$$\delta(d, y^E) < 1 \;\Leftrightarrow\; d + (1-m)y^E < \overline{z} - \underline{z} \;\Leftrightarrow\; y^E < \frac{\overline{z} - \underline{z} - d}{1-m} = y^E_{ub}(d). \tag{B4}$$

The upper end $y^E_{ub}(d)$ is exactly the level of y^E at which $\delta = 1$.

Consider first $d \in [0, d_c(m))$. By Steps 1-A and 1-B, the root $y^E(d)$ lies in the interior of the feasible range, whose upper end is $\min\{y^E_h,\ y^E_{ub}(d)\} \le y^E_{ub}(d)$, so $y^E(d) < y^E_{ub}(d)$. Hence $\delta(d, y^E(d)) < 1$ by (B4).

Consider next $d \to d_c(m)$. By the definition of $d_c(m)$, $G(d_c(m), y^E_{ub}(d_c(m))) = 0$, and by the strict monotonicity of Step 1-B, $y^E_{ub}(d_c(m))$ is the unique root at $d = d_c(m)$. Since $\partial G/\partial y^E < 0$ implies

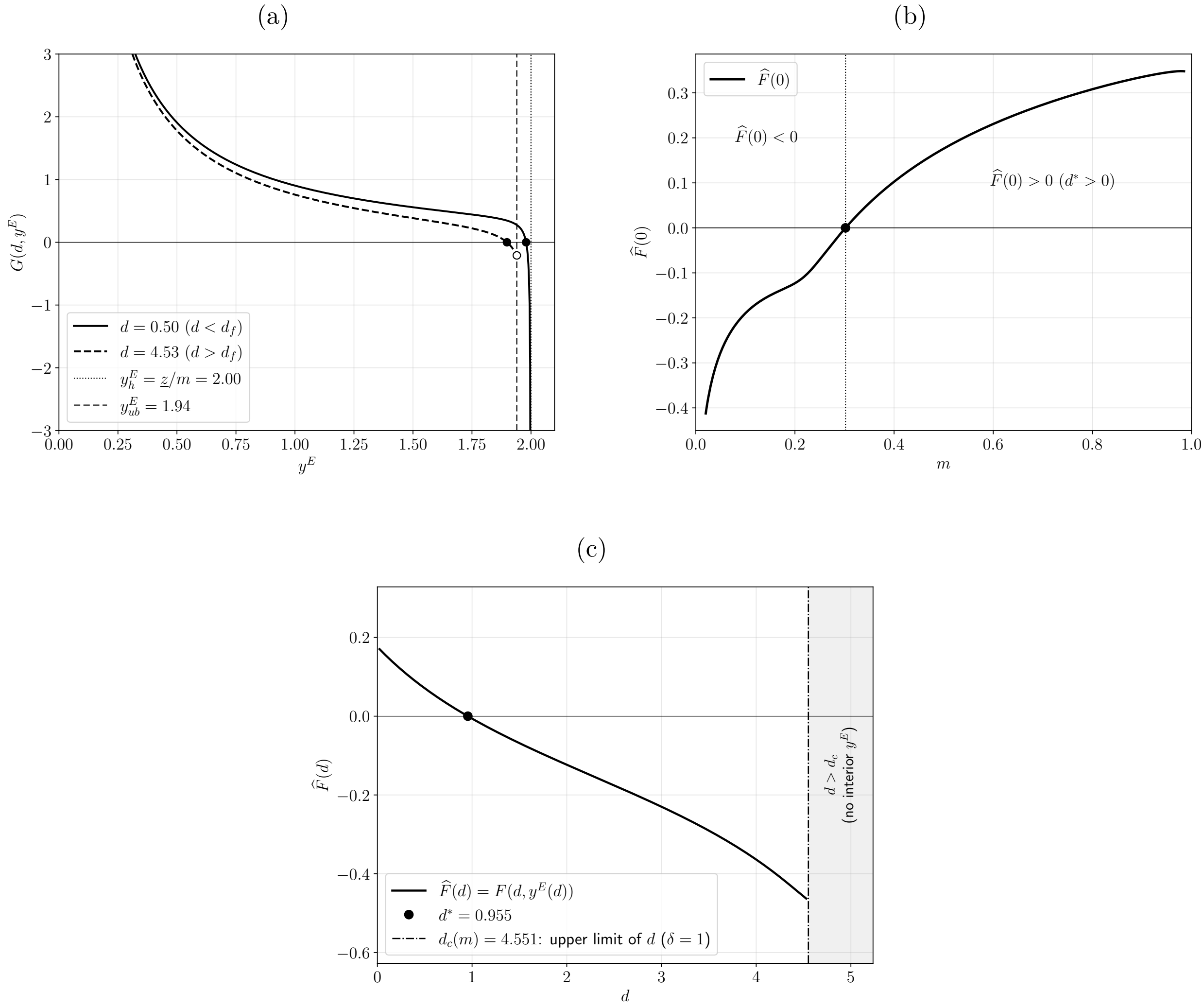

Figure 9: Numerical illustrations for the proof of Proposition 1 (baseline: $\underline{z} = 1$, $\overline{z} = 6.5$, $z_1 = 2.5$, $\alpha = 0.03$). Panel (a) corresponds to Step 1, panels (b) and (c) to Step 2. See the corresponding Steps in the text for details of each panel.

that $y^E(d)$ is continuous in d, we have $y^E(d) \to y^E_{ub}(d_c(m))$ as $d \uparrow d_c(m)$, and hence $\delta(d, y^E(d)) \to \delta(d_c(m), y^E_{ub}(d_c(m))) = 1$.

$\square$

Step 2: The borrowing shutdown level and existence of equilibrium

By Lemma 1, for every $d \in [0, \overline{z} - \underline{z})$ there is a unique $y^E(d) \in \Omega$ satisfying (20). Define the function obtained by substituting $y^E(d)$ for y^E in (19) as

$$\begin{aligned}\widehat{F}(d) &\equiv F(d, y^E(d))\\ &= \frac{(1+\alpha)Q_m(d, y^E(d))}{z_1 + q(d, y^E(d))d} + p_z \ln m + \alpha p_z \left[\ln X_d(y^E(d)) - \ln X_r(\overline{z}, d, y^E(d))\right]\end{aligned}$$

$$= 0. \tag{B5}$$

An equilibrium is a d^* satisfying $\widehat{F}(d^*) = 0$.

We first consider the conditions for an equilibrium with $d^* > 0$ to exist. The following lemma holds.

Lemma 2. *Suppose Assumption 1 holds and z_1 is sufficiently small. Then there exists $m \in (0,1)$ satisfying $\widehat{F}(0) = 0$; define the smallest such m as the borrowing shutdown level $\underline{m}(\alpha) \in (0,1)$. Then $\widehat{F}(0) < 0$ for $m < \underline{m}(\alpha)$, $d^* = 0$ at $m = \underline{m}(\alpha)$, and $d^* > 0$ in a right neighborhood of $\underline{m}(\alpha)$.*

Proof of Lemma 2.

Recall that $\widehat{F}(d) = 0$ is the first-order condition of the optimization. To define the borrowing shutdown level $\underline{m}(\alpha)$, view $\widehat{F}(0)$ as a function of m. Evaluating (B5) at $d = 0$ gives

$$\widehat{F}(0) = \frac{(1+\alpha)\left[1 - p_z(1-m)y^E(0)\right]}{z_1} + p_z \ln m + \alpha p_z \left[\ln(\underline{z} - my^E(0)) - \ln(\overline{z} - y^E(0))\right], \tag{B6}$$

where $y^E(0)$ satisfies (20) evaluated at $d = 0$,

$$G(0, y^E(0)) = (1-m)p_z \ln m + \frac{1}{y^E(0)} - \frac{\alpha m(1-m)p_z y^E(0)}{\underline{z} - my^E(0)} + \alpha p_z\left[\ln(\underline{z} - my^E(0)) - \ln(\overline{z} - y^E(0))\right]$$
$$= 0, \tag{B7}$$

which does not contain z_1. Hence $y^E(0)$ does not depend on z_1.

We show below that when z_1 is sufficiently small, (i) $\lim_{m\to 0} \widehat{F}(0) < 0$ and (ii) $\lim_{m\to 1} \widehat{F}(0) > 0$. Together with the continuity of $\widehat{F}(0)$ in m, these imply that an m satisfying $\widehat{F}(0) = 0$ exists, and the borrowing shutdown level $\underline{m}(\alpha)$ can be defined from it.

Consider first the case $m \to 0$. Taking the limit $m \to 0$ in (B7), the third and subsequent terms all converge to constants while the first term diverges to $-\infty$. For the equation to balance, $1/y^E(0)$ must adjust accordingly, so

$$\lim_{m\to 0} y^E(0) = 0.$$

With this in mind, (B6) implies

$$\lim_{m\to 0}\widehat{F}(0) = \frac{1+\alpha}{z_1} + p_z\left(\lim_{m\to 0}\ln m\right) + \alpha p_z\left(\ln\underline{z} - \ln\overline{z}\right)$$
$$= -\infty$$
$$< 0. \tag{B8}$$

Hence (i) holds.

Next, noting that $y^E(0)$ does not depend on z_1,

$$\lim_{z_1\to 0}\left[\lim_{m\to 1}\widehat{F}(0)\right] = \lim_{z_1\to 0}\left[\frac{1+\alpha}{z_1} + \alpha p_z\left[\ln(\underline{z} - \lim_{m\to 1} y^E(0)) - \ln(\overline{z} - \lim_{m\to 1} y^E(0))\right]\right]$$
$$= \infty > 0. \tag{B9}$$

Hence (ii) holds if z_1 is sufficiently small.

By (B8), (B9), and the continuity of $\widehat{F}(0)$ in m, the intermediate value theorem yields at least one $m \in (0,1)$ satisfying $\widehat{F}(0) = 0$. Let $\underline{m}(\alpha)$ be the smallest such m.

By definition, $\widehat{F}(0) = 0$ at $m = \underline{m}(\alpha)$. Since $\underline{m}(\alpha)$ is the smallest root of $\widehat{F}(0)$, no zero exists on the interval $(0, \underline{m}(\alpha))$. On this interval $\widehat{F}(0)$ is continuous, has no zero, and is negative for sufficiently small m by (B8), so $\widehat{F}(0) < 0$ on all of $(0, \underline{m}(\alpha))$. Hence $\widehat{F}(0)$ turns from negative to positive at $m = \underline{m}(\alpha)$, and $\widehat{F}(0) > 0$ in its right neighborhood (see Figure 9(b)).

Figure 9(b) illustrates this numerically. Plotting $\widehat{F}(0)$ as a function of m under the baseline parameters, the smallest root of $\widehat{F}(0) = 0$ gives the borrowing shutdown level $\underline{m}(\alpha)$. In this example the root is unique, $\underline{m}(0.03) \simeq 0.30$ (vertical dotted line), and its $\alpha \to 0$ limit is $\underline{m}(0) \simeq 0.29$.

The condition on z_1 in this lemma is a sufficient, not a necessary, condition for the existence of an m with $\widehat{F}(0) > 0$. Indeed, in the $\alpha \to 0$ limit treated in Appendix C, the borrowing shutdown level exists uniquely for any $z_1 > 0$ ((C5)).

Finally, we show that equilibrium debt issuance d^* exists with $d^* \geq 0$ at $m = \underline{m}(\alpha)$ and in its right neighborhood. First, by the definition of $\underline{m}(\alpha)$, $d^* = 0$ at $m = \underline{m}(\alpha)$.

Consider next $m > \underline{m}(\alpha)$. Recall that $\widehat{F}(d)$ is the first-order condition for d with $y^E = y^E(d)$ kept optimal, and that $\widehat{F}(d^*) = 0$ holds in equilibrium. As shown in Step 3 below (Lemma 3), at an interior point d^* where $\widehat{F}$ turns from positive to negative ($\widehat{F}'(d^*) < 0$), the second-order conditions are satisfied and d^* is a maximum of the objective. It therefore suffices to show that $\widehat{F}(0) > 0$ while

$\widehat{F}(d) < 0$ for sufficiently large d; a maximum $d^* > 0$ then obtains at an interior point in between.

By Lemma 1, along the curve $y^E(d)$ on which y^E is kept optimal, d ranges over $[0, d_c(m))$, and as $d \uparrow d_c(m)$ the curve reaches the certain-default boundary $\delta = 1$. Consider $d \to d_c(m)$.

By the definition of $d_c(m)$, $\delta(d, y^E(d)) \to 1$ (certain default) as $d \uparrow d_c(m)$. Hence

$$\lim_{d \uparrow d_c(m)} \delta(d, y^E(d)) = 1, \quad \lim_{d \uparrow d_c(m)} q(d, y^E(d)) = 0,$$

$$\lim_{d \uparrow d_c(m)} Q_m(d, y^E(d)) = -p_z d_c(m), \quad \lim_{d \uparrow d_c(m)} \left[X_r(\overline{z}, d, y^E(d)) - X_d(y^E(d))\right] = 0,$$

where the last relation follows from (B3). Substituting these into (B5) gives

$$\lim_{d \uparrow d_c(m)} \widehat{F}(d) = -\frac{(1+\alpha) p_z d_c(m)}{z_1} + p_z \ln m \ < \ 0.$$

Therefore, for m with $\widehat{F}(0) > 0$, combining $\lim_{d \uparrow d_c(m)} \widehat{F}(d) < 0$ with the continuity of $\widehat{F}$ in d, the intermediate value theorem yields a $d^* \in (0, d_c(m))$ satisfying $\widehat{F}(d^*) = 0$ (see Figure 9(c)).

Figure 9(c) plots $\widehat{F}(d) = F(d, y^E(d))$ over $d \in [0, d_c(m))$ under $m = 0.5$. As $d \to 0$, $\widehat{F} > 0$, while at the certain-default boundary $d_c(m) \simeq 4.55$ ($\delta = 1$, the upper limit of d), $\widehat{F} < 0$, so a unique d^* satisfying $\widehat{F}(d^*) = 0$ exists in between (filled circle). No interior solution exists for $d > d_c(m)$ (shaded). Since $\widehat{F}$ crosses 0 from above at the intersection, the second-order conditions are also satisfied by Lemma 3.

Note that $\widehat{F}(0) > 0$ is a sufficient, not a necessary, condition for $d^* > 0$: even with $\widehat{F}(0) < 0$, the d maximizing V_I may remain interior. When α is sufficiently small, it can be shown that in the $\alpha \to 0$ limit, debt issuance is positive if and only if $m > \underline{m}(0)$ (Appendix C).

In sum, at each point in the right neighborhood of $\underline{m}(\alpha)$ there exist $d^* > 0$ and the corresponding $y^{E*} = y^E(d^*)$, while $d^* = 0$ at $m = \underline{m}(\alpha)$. The analysis hereafter focuses on the range $m \in [\underline{m}(\alpha), 1]$ over which debt issuance is non-negative ($d^* \geq 0$).

$\square$

Step 3: Verifying the second-order conditions

Steps 1 and 2 established that a point $(d^*, y^E(d^*))$ satisfying $F(d^*, y^E(d^*)) = 0$ and $G(d^*, y^E(d^*)) = 0$ exists in the interior of the feasible range. We now show that the second-order conditions of the optimization are satisfied at this point.

First, we state the second-order conditions. With $F = \partial V_I / \partial d$ and $G = \partial V_I / \partial y^E$, the Hessian of

V_I with respect to (d, y^E) is

$$H = \begin{pmatrix} \dfrac{\partial^2 V_I}{\partial d^2} & \dfrac{\partial^2 V_I}{\partial d \partial y^E} \\ \dfrac{\partial^2 V_I}{\partial y^E \partial d} & \dfrac{\partial^2 V_I}{\partial (y^E)^2} \end{pmatrix} = \begin{pmatrix} F_d & F_y \\ G_d & G_y \end{pmatrix} = \begin{pmatrix} F_d & G_d \\ G_d & G_y \end{pmatrix}, \tag{B10}$$

where the last equality uses $F_y = G_d$ by Young's theorem. The second-order condition for $(d^*, y^E(d^*))$ to be a local maximum of V_I is that the Hessian H be negative definite, that is,

$$F_d < 0, \tag{B11}$$

$$G_y < 0, \tag{B12}$$

$$\det(H) \equiv F_d G_y - G_d^2 > 0. \tag{B13}$$

We show below that (B11)–(B13) are satisfied at this $(d^*, y^E(d^*))$.

Lemma 3. *Suppose all the conditions of Proposition 1 hold. Then, except for non-generic degenerate cases, there exists d^* (with $(d^*, y^E(d^*)) \in \Omega$) satisfying $\widehat{F}(d^*) = F(d^*, y^E(d^*)) = 0$ and $\widehat{F}'(d^*) < 0$, and at such d^* the second-order conditions* (B11)*–*(B13) *are all satisfied.*

Proof of Lemma 3. Applying the implicit function theorem to $G(d, y^E(d)) = 0$ gives $dy^E(d)/dd = -G_d/G_y$. Combining this with Young's theorem, $F_y = G_d$,

$$\widehat{F}'(d) = F_d + F_y \cdot \frac{dy^E(d)}{dd} = F_d - \frac{G_d^2}{G_y} = \frac{\det(H)}{G_y}. \tag{B14}$$

By Step 2, $\widehat{F}$ crosses 0 from above at least once (see Figure 9(c)). At such a crossing d^*, $\widehat{F}'(d^*) \leq 0$; the degenerate case $\widehat{F}'(d^*) = 0$ has measure zero in the parameter space and is ignored, so we focus on d^* with $\widehat{F}'(d^*) < 0$.

By Lemma 1, $G_y < 0$; combining this with (B14) and $\widehat{F}'(d^*) < 0$ gives $\det(H) = \widehat{F}'(d^*) \cdot G_y > 0$, so (B13) holds. Together with $G_y < 0$, this gives $F_d G_y > G_d^2 \geq 0 \Rightarrow F_d < 0$, so (B11) holds. (B12) is $G_y < 0$ itself. $\square$

Proof of Proposition 1. Combining Lemma 1, Lemma 2, and Lemma 3 yields the claim of Proposition 1. $\blacksquare$

C Proof of Proposition 2

This Appendix analyzes the equilibrium in the limit $\alpha \to 0$ and examines the relationship between the strength of entitlement protection m and the equilibrium default probability δ^* when α is sufficiently small. All evaluations of partial derivatives, limits, and monotonicity of V_I are carried out in the interior of the feasible set (21),

$$\Omega \equiv \left\{(d, y^E) \ : \ 0 \le y^E < y_h^E, \ \ 0 \le d < d_h(y^E)\right\}, \quad y_h^E = \frac{\underline{z}}{m}, \ \ d_h(y^E) = \overline{z} - \underline{z} - (1-m)y^E.$$

The upper bound on d, $d_h(y^E)$, is a decreasing function of y^E, and at the corner-solution limit $y^E \to y_h^E = \underline{z}/m$ treated below it takes the value $d_h(y_h^E) = d_f = \overline{z} - \underline{z}/m$ ((B1)). On Ω, $X_r(\overline{z}, d, y^E) = \overline{z} - d - y^E > X_d(y^E) > 0$ always holds. Note that Ω is determined by $(\overline{z}, \underline{z}, m)$ alone and does not depend on α, so taking the limit $\alpha \to 0$ leaves the range of (d, y^E) unchanged, and the limit and monotonicity arguments can be carried out consistently on a fixed domain.

The proof in this Appendix rests on two lemmas. First, Lemma 4 determines (d^*, y^{E*}) as $\alpha \to 0$. Next, Lemma 6 shows that in the limit $\alpha \to 0$ and $m \to \underline{m}(0)$, the default probability δ^* is decreasing in m. Finally, these two lemmas and continuity yield Proposition 2. The $\underline{m}(0)$ appearing below is the borrowing shutdown level in the $\alpha \to 0$ limit; its definition and uniqueness are given in the proof of Lemma 4, immediately after (C5). Between the two lemmas we insert a lemma (Lemma 5) justifying the interchange of limits and derivatives, which is also used in the comparative statics of the subsequent Propositions.

Lemma 4. *Suppose Assumption 1 holds and $m \in (\underline{m}(0), 1)$ is arbitrarily fixed. Let*

$$d_0^*(m) = \frac{2 - \ln m \left[1 - p_z(1-m) \cdot \dfrac{\underline{z}}{m}\right] - \sqrt{4 + (\ln m)^2 \left[1 - p_z(1-m) \cdot \dfrac{\underline{z}}{m}\right]^2 + 4p_z(\ln m)^2 z_1}}{-2p_z \ln m}.$$

Suppose z_1 satisfies the conditions

$$\text{(CL4a)}\ z_1 \le z_1^{max}(m) \equiv \frac{1 - p_z(1-m) \cdot \dfrac{\underline{z}}{m}}{p_z(-\ln m)},$$

$$\text{(CL4b)}\ \frac{d_0^*(m)}{z_1 + q(d_0^*(m), \underline{z}/m)d_0^*(m)} \le \frac{m}{(1-m)p_z\underline{z}} + \ln m.$$

Then, in the limit $\alpha \to 0$, the equilibrium values of debt issuance and entitlement setting converge to

$$\lim_{\alpha \to 0} y^{E*}(\alpha, m) = y_0^{E*}(m) = \frac{\underline{z}}{m},$$

$$\lim_{\alpha \to 0} d^*(\alpha, m) = d_0^*(m) \geq 0.$$

Proof of Lemma 4. Fix $m \in (\underline{m}(0), 1)$. We conjecture that in the $\alpha \to 0$ limit the optimal y^E converges to the corner $\underline{z}/m$, and verify that this conjecture satisfies the first-order conditions (19) and (20). Compactness then shows that the $\alpha > 0$ equilibrium converges to this corner solution.

Step 4(i). The equilibrium d_0^* corresponding to the corner $y^E = \underline{z}/m$ and condition (CL4a).

Denote by $\{d_0^*(m),\ y_0^{E*}(m)\}$ the debt issuance and entitlement setting in the limit $\alpha \to 0$. First, conjecture $y_0^{E*}(m) = \underline{z}/m$. Then the first-order condition for d, (19), gives

$$\begin{aligned}\lim_{\alpha \to 0} \frac{\partial V_I}{\partial d}(d, y^E; m, \alpha) &= \lim_{\alpha \to 0} F(d, y^E; m, \alpha) \\ &= \frac{Q_m(d, \underline{z}/m)}{z_1 + q(d, \underline{z}/m)d} + p_z \ln m = 0. \qquad \text{(C1)}\end{aligned}$$

The d satisfying this equation is $d_0^*(m)$. Clearing the denominator, (C1) can be arranged into the following quadratic equation in d_0^*:

$$\begin{aligned}p_z^2(-\ln m)(d_0^*(m))^2 - p_z\big[2 + (-\ln m)\big(1 - p_z(1-m)\underline{z}/m\big)\big]d_0^*(m) \\ + \big[1 - p_z(1-m)\underline{z}/m - p_z(-\ln m)z_1\big] = 0. \qquad \text{(C2)}\end{aligned}$$

This quadratic opens upward, and of its two roots, the one satisfying the second-order condition and corresponding to a local maximum of V_I is the smaller positive root. Hence, provided the conjecture $\lim_{\alpha \to 0} y^{E*}(m, \alpha) = y_0^{E*}(m) = \underline{z}/m$ is correct, debt issuance in the $\alpha \to 0$ limit is

$$\begin{aligned}&\lim_{\alpha \to 0} d^*(m, \alpha) = d_0^*(m) \\ &= \frac{2 - \ln m\left[1 - p_z(1-m)\cdot\frac{\underline{z}}{m}\right] - \sqrt{4 + (\ln m)^2\left[1 - p_z(1-m)\cdot\frac{\underline{z}}{m}\right]^2 + 4p_z(\ln m)^2 z_1}}{-2p_z \ln m} \qquad \text{(C3)}\end{aligned}$$

The condition for debt issuance to be non-negative in the $\alpha \to 0$ limit follows from (C3) as

$$d_0^*(m) \geq 0 \ \Leftrightarrow\ z_1 \leq z_1^{max}(m) \equiv \frac{1 - p_z(1-m)\cdot\frac{\underline{z}}{m}}{p_z(-\ln m)}. \qquad \text{(C4)}$$

This is condition (CL4a) of Lemma 4.

Writing the constant term of (C2) as

$$\varphi(m) \equiv 1 - p_z(1-m)\frac{\underline{z}}{m} - p_z(-\ln m)z_1, \tag{C5}$$

we see that $z_1 \leq z_1^{max}(m)$ and $\varphi(m) \geq 0$ are equivalent, so (C4) can be restated as $d_0^*(m) \geq 0 \Leftrightarrow \varphi(m) \geq 0$. Since $\varphi'(m) = p_z\underline{z}/m^2 + p_z z_1/m > 0$, φ is strictly increasing in m (equivalently, $z_1^{max}(m)$ in (C4) is increasing in m). Moreover, $\lim_{m\to 0}\varphi(m) = -\infty$ and $\varphi(1) = 1 > 0$, so a unique $m \in (0,1)$ satisfying $\varphi(m) = 0$ exists; write it as $\underline{m}(0)$. Since the first-order coefficient of (C2) is positive for $m \geq \underline{m}(0)$, moreover, its smaller positive root satisfies

$$d_0^*(m) > 0 \;\Leftrightarrow\; \varphi(m) > 0 \;\Leftrightarrow\; m > \underline{m}(0).$$

That is, in the $\alpha \to 0$ limit, debt issuance is positive if and only if m exceeds $\underline{m}(0)$, and $d_0^*\big(\underline{m}(0)\big) = 0$ at $m = \underline{m}(0)$. This characterization holds regardless of the size of z_1.

Step 4(ii). The optimal y^E given $d = d_0^*(m)$ and condition (CL4b).

Step 4(i) confirmed that if $y_0^{E*}(m) = \underline{z}/m$, then $d_0^*(m)$ is given by (C3). We now show the converse: if $d_0^*(m)$ is given as in (C3), then $y_0^{E*}(m) = \underline{z}/m$ obtains.

Conjecture that $d_0^*(m)$ is given by (C3). Then the first-order condition for y^E, (20), gives, for any $y^E \in \Omega^y$,

$$\begin{aligned}\lim_{\alpha\to 0}\frac{\partial V_I}{\partial y^E}(d, y^E; m, \alpha) &= \lim_{\alpha\to 0} G(d_0^*(m), y^E; m, \alpha)\\ &= -\frac{(1-m)p_z d_0^*(m)}{z_1 + q(d_0^*(m), y^E)d_0^*(m)} + (1-m)p_z\ln m + \frac{1}{y^E}.\end{aligned} \tag{C6}$$

If the right-hand side of (C6) is positive for every $y^E \in \Omega^y$, then y^E hits its upper bound and $y_0^{E*}(m) = \underline{z}/m$ obtains in the $\alpha \to 0$ limit. We now derive the condition for the right-hand side of (C6) to be positive for every $y^E \in \Omega^y$.

First, partially differentiating the right-hand side of (C6) with respect to y^E gives

$$\frac{\partial}{\partial y^E}\left[\lim_{\alpha\to 0}\frac{\partial V_I}{\partial y^E}\right] \;=\; -\frac{[(1-m)p_z d_0^*(m)]^2}{[z_1 + q(d_0^*(m), y^E)d_0^*(m)]^2} \;-\; \frac{1}{(y^E)^2} \;<\; 0,$$

so the right-hand side of (C6) is strictly decreasing in y^E.

Hence a necessary and sufficient condition for the right-hand side of (C6) to be positive for every

$y^E \in \Omega^y$ is that its limit at the upper bound $y^E \to \underline{z}/m$ be non-negative; that is,

$$\lim_{y^E \uparrow \underline{z}/m} \lim_{\alpha \to 0} \left.\frac{\partial V_I}{\partial y^E}\right|_{d=d_0^*(m)} = \frac{m}{\underline{z}} - (1-m)p_z \left[\frac{d_0^*(m)}{z_1 + q(d_0^*(m), \underline{z}/m) d_0^*(m)} - \ln m\right] \geq 0. \tag{C7}$$

When (C7) holds, V_I in the limit problem with $d = d_0^*(m)$ fixed is strictly increasing in y^E, and the optimal y^E is $\lim_{\alpha \to 0} y^{E*} = \underline{z}/m$. Substituting (C3) into (C7) and rearranging, the condition becomes

$$\frac{d_0^*(m)}{z_1 + q(d_0^*(m), \underline{z}/m) d_0^*(m)} \leq \frac{m}{(1-m)p_z \underline{z}} + \ln m. \tag{C8}$$

This is condition (CL4b) of Lemma 4.

In sum, when (C4) and (C8) hold, in the limit $\alpha \to 0$ the equilibrium values of debt issuance and entitlement setting converge to

$$\lim_{\alpha \to 0} y^{E*}(\alpha, m) = y_0^{E*}(m) = \frac{\underline{z}}{m},$$

$$\lim_{\alpha \to 0} d^*(\alpha, m) = d_0^*(m).$$

$\square$

The comparative statics of the subsequent Propositions use the interchange of the m-derivative of the equilibrium quantities (the comparative statics) with the $\alpha \to 0$ limit. The next lemma justifies this by computing the comparative statics explicitly via the implicit function theorem at fixed $\alpha > 0$ and then actually taking the limit $\alpha \to 0$.

In what follows, let $\delta_0(m)$ denote the default probability in the $\alpha \to 0$ limit, $p_z[d_0^*(m) + (1-m)\underline{z}/m]$.

Lemma 5. *Suppose all the conditions of Lemma 4 hold (Assumption 1, $m \in (\underline{m}(0), 1)$, and conditions (CL4a) and (CL4b) on z_1). Suppose further that*

$$a_0(m) \equiv \frac{m}{\underline{z}} - (1-m)p_z \left[\frac{d_0^*(m)}{z_1 + q(d_0^*(m), \underline{z}/m) d_0^*(m)} - \ln m\right],$$

given by the left-hand side of (C7)*, satisfies $a_0(m) \neq 0$. Then the comparative statics of the interior solution satisfy, as $\alpha \to 0$,*

$$\lim_{\alpha \to 0} \frac{dd^*}{dm} = \frac{dd_0^*}{dm}, \qquad \lim_{\alpha \to 0} \frac{dy^{E*}}{dm} = -\frac{\underline{z}}{m^2},$$

and consequently $\lim_{\alpha \to 0} \dfrac{d\delta^*}{dm} = \dfrac{d\delta_0(m)}{dm}$.

Proof of Lemma 5. The proof proceeds in three steps. Step 5(i) reduces the comparative statics to one dimension via the implicit function theorem at fixed $\alpha > 0$; Step 5(ii) derives the $\alpha \to 0$ limits of the divergent terms and of the ratios G_m/G_y and G_d/G_y; and Step 5(iii) combines these to obtain the limits of the comparative statics.

Step 5(i) (Comparative statics at fixed α: reduction to one dimension).

By (B5), equilibrium debt issuance is characterized by $\widehat{F}(d^*) = 0$. Applying the implicit function theorem to $G(d, y^E(d)) = 0$ from (20) and to $\widehat{F}(d^*) = 0$ gives

$$\frac{dd^*}{dm} = -\frac{\widehat{F}_m}{\widehat{F}_d}, \qquad \widehat{F}_d = F_d - F_y\frac{G_d}{G_y} = \frac{\det H}{G_y}, \qquad \widehat{F}_m = F_m - F_y\frac{G_m}{G_y}, \tag{C9}$$

$$\frac{dy^{E*}}{dm} = -\frac{G_d}{G_y}\frac{dd^*}{dm} - \frac{G_m}{G_y} \tag{C10}$$

where $F_y = G_d$ by Young's theorem and $\det H = F_dG_y - G_d^2$ by (B10). These hold exactly for any $\alpha > 0$. Moreover, $G_y < 0$ by (B12), and $\widehat{F}_d < 0$ by (B14), (B12), and (B13). We now take $\alpha \to 0$ and evaluate these quantities in the limit.

Step 5(ii) (Limits of the divergent terms and of the ratios G_m/G_y and G_d/G_y).

We first derive the limit of $\alpha\delta m/X_d$, which is used repeatedly below. Solving (20) ($\partial V_I/\partial y^E = 0$) for $\alpha\delta m/X_d$ gives

$$\frac{\alpha\delta m}{X_d} = a(\alpha) + \alpha p_z \ln\frac{X_d}{X_r}, \quad a(\alpha) \equiv -\frac{(1+\alpha)(1-m)p_z d}{z_1 + qd} + (1-m)p_z\ln m + \frac{1}{y^E}. \tag{C11}$$

Since conditions (CL4a) and (CL4b) of Lemma 4 hold, $d \to d_0^*(m)$ and $y^E \to \underline{z}/m$ as $\alpha \to 0$, so $a(\alpha)$ converges to the finite value

$$\lim_{\alpha\to 0} a(\alpha) = \frac{m}{\underline{z}} - (1-m)p_z\left[\frac{d_0^*}{z_1 + q(d_0^*, \underline{z}/m)d_0^*} - \ln m\right] = a_0(m).$$

That is, the limit of $a(\alpha)$ coincides with the $a_0(m)$ defined in the statement (the left-hand side of (C7)). By condition (CL4b) ((C7)), $a_0(m) \geq 0$, so under the assumption $a_0(m) \neq 0$ of the statement, $a_0(m) > 0$.

We now show from (C11) that $\alpha\delta m/X_d$ converges to $a_0(m)$. As $\alpha \to 0$, $X_d = \underline{z} - my^{E*} \to 0$ while $X_r = \overline{z} - d - y^E \to \overline{z} - d_0^* - \underline{z}/m > 0$ stays positive, so for sufficiently small α, $X_d < X_r$, that is,

$\ln(X_d/X_r) < 0$. Hence, for sufficiently small α,

$$\frac{\alpha\delta m}{X_d} = a(\alpha) + \alpha p_z \ln\frac{X_d}{X_r} < a(\alpha).$$

Since $a(\alpha) \to a_0(m)$ is finite as $\alpha \to 0$, for sufficiently small α, $a(\alpha)$ itself is bounded above by some fixed finite constant. Combined with the inequality above, $\alpha\delta m/X_d$ is therefore bounded above, for all sufficiently small α, by a finite constant independent of α. Using this, we show $\alpha \ln X_d \to 0$ by contradiction.

Suppose $\alpha \ln X_d \not\to 0$ as $\alpha \to 0$. Since $X_d \to 0$, we have $\alpha \ln X_d < 0$ for sufficiently small α. That $\alpha \ln X_d$ fails to converge to 0 as $\alpha \to 0$ therefore means that for some $\eta > 0$ and some sequence $\alpha_n \to 0$,

$$\alpha_n \ln X_d \leq -\eta \;\Leftrightarrow\; X_d \leq e^{-\eta/\alpha_n} \tag{C12}$$

holds along that sequence.

Since we are inside the feasible range, $d \geq 0$, so $\delta m = p_z\big[d + (1-m)y^E\big]m \geq p_z(1-m)y^E m$. Moreover, since $y^E \to \underline{z}/m > 0$ as $\alpha \to 0$, we have $y^E > \underline{z}/(2m)$ for α close enough to zero. Hence

$$\delta m \geq p_z(1-m)y^E m > \frac{p_z(1-m)\underline{z}}{2} \equiv b > 0, \tag{C13}$$

where $b > 0$ is a constant independent of α. By (C12) and (C13), along the sequence α_n,

$$\frac{\alpha_n\delta m}{X_d} \;\geq\; \frac{\alpha_n b}{e^{-\eta/\alpha_n}} \;=\; b\alpha_n e^{\eta/\alpha_n}.$$

As $\alpha_n \to 0$, the exponential e^{η/α_n} diverges faster than α_n converges to zero, so this implies

$$\frac{\alpha_n\delta m}{X_d} \to \infty,$$

contradicting the boundedness of $\alpha_n\delta m/X_d$. Hence $\alpha \ln X_d \to 0$ must hold.

Taking limits in (C11) therefore gives

$$\lim_{\alpha\to 0}\frac{\alpha\delta m}{X_d} = a_0(m). \tag{C14}$$

Consider next the limits of G_m/G_y and G_d/G_y as $\alpha \to 0$. From (B2),

$$G_y = -\frac{(1+\alpha)(1-m)^2 p_z^2 d^2}{(z_1+qd)^2} - \frac{1}{(y^E)^2} - \frac{\alpha\delta m^2}{X_d^2} + \alpha p_z\Big[\frac{1}{X_r} - \frac{m(2-m)}{X_d}\Big].$$

Splitting this into the divergent term $-\alpha\delta m^2/X_d^2$ and the remainder R_y, and likewise splitting $G_m = \partial G/\partial m$ into the divergent term $-\alpha\delta m y^E/X_d^2$ and the remainder R_m, we have

$$G_y = -\frac{\alpha\delta m^2}{X_d^2} + R_y, \quad G_m = -\frac{\alpha\delta m y^E}{X_d^2} + R_m,$$

where

$$\begin{aligned} R_y &\equiv -\frac{(1+\alpha)(1-m)^2 p_z^2 d^2}{(z_1+qd)^2} - \frac{1}{(y^E)^2} + \alpha p_z\Big[\frac{1}{X_r} - \frac{m(2-m)}{X_d}\Big], \\ R_m &\equiv \frac{(1+\alpha)p_z d}{z_1+qd} + \frac{(1+\alpha)(1-m)p_z^2 y^E d^2}{(z_1+qd)^2} - p_z \ln m + \frac{(1-m)p_z}{m} + \frac{\alpha(p_z y^E m - \delta)}{X_d} - \frac{\alpha p_z y^E}{X_d}. \end{aligned}$$

By (C14), $\alpha\delta m/X_d \to a_0(m)$ as $\alpha \to 0$, so $X_d/\alpha \to \delta_0(m)m/a_0(m) > 0$; hence α/X_d converges to a positive finite value, and both R_y and R_m converge to finite values.

Noting that as $\alpha \to 0$ we have $X_d \to 0$ and $\alpha\delta m/X_d \to a_0(m) > 0$, that is, $X_d^2/(\alpha\delta m) \to 0$, the common factor $-\alpha\delta m/X_d^2$ cancels between numerator and denominator, yielding

$$\lim_{\alpha\to 0}\frac{G_m}{G_y} = \lim_{\alpha\to 0}\frac{-\dfrac{\alpha\delta m}{X_d^2}\Big(y^E - \dfrac{R_m X_d^2}{\alpha\delta m}\Big)}{-\dfrac{\alpha\delta m}{X_d^2}\Big(m - \dfrac{R_y X_d^2}{\alpha\delta m}\Big)} = \lim_{\alpha\to 0}\frac{y^E - \dfrac{R_m X_d^2}{\alpha\delta m}}{m - \dfrac{R_y X_d^2}{\alpha\delta m}} = \frac{y_0^E}{m} = \frac{\underline{z}}{m^2}. \tag{C15}$$

Consider next G_d/G_y. Writing out $G_d = \partial G/\partial d$ and noting that X_d does not depend on d,

$$G_d = -\frac{(1+\alpha)(1-m)p_z(z_1+p_z d^2)}{(z_1+qd)^2} + \frac{\alpha p_z}{X_r} - \frac{\alpha p_z m}{X_d}.$$

Noting that $d \to d_0^*(m)$ and $y^{E*} \to \underline{z}/m$ as $\alpha \to 0$, and that $\alpha\delta m/X_d \to a_0(m)$ (hence $\alpha/X_d \to a_0(m)/(\delta_0(m)m)$) by (C14), every term of G_d converges to a finite value. Hence G_d converges to a finite value as $\alpha \to 0$.

As for the behavior of G_y: as argued above, G_y is dominated by its leading term $-\alpha\delta m^2/X_d^2$. Since $\alpha/X_d^2 = (\alpha/X_d)\cdot(1/X_d) \to +\infty$ and $\delta \to \delta_0(m) > 0$, this leading term diverges to $-\infty$ and cannot be offset by the bounded R_y, so $G_y \to -\infty$.

Since a finite-limit G_d is divided by a G_y whose absolute value diverges,

$$\lim_{\alpha\to 0}\frac{G_d}{G_y}=0. \tag{C16}$$

Step 5(iii) (Limits of the comparative statics).

Consider next $\{F_d, F_y, F_m\}$. From (19) and $Q_m=\partial(qd)/\partial d=1-p_z(1-m)y^E-2p_zd$ ((16)),

$$\begin{aligned}
F_d &= -(1+\alpha)\frac{2p_z(z_1+qd)+Q_m^2}{(z_1+qd)^2}+\frac{\alpha p_z}{X_r},\\
F_y &= -(1+\alpha)\frac{p_z(1-m)(z_1+p_zd^2)}{(z_1+qd)^2}-\frac{\alpha p_z m}{X_d}+\frac{\alpha p_z}{X_r},\\
F_m &= (1+\alpha)\frac{p_z y^E(z_1+p_zd^2)}{(z_1+qd)^2}+\frac{p_z}{m}-\frac{\alpha p_z y^E}{X_d},
\end{aligned}$$

where the relation $Q_m=q-p_zd$ is used in the first terms of F_y and F_m.

Since $d\to d_0^*(m)$ and $y^{E*}\to \underline{z}/m$ as $\alpha\to 0$, and $\alpha\delta m/X_d\to a_0(m)$ by (C14), all of $\{F_d, F_y, F_m\}$ converge to finite values as $\alpha\to 0$; denote these limits by F_{d0}, F_{y0}, F_{m0}. Then, from (C9), (C15), and (C16),

$$\begin{aligned}
&\lim_{\alpha\to 0}\widehat{F}_d = F_{d0}\neq 0,\\
&\lim_{\alpha\to 0}\widehat{F}_m = F_{m0}-F_{y0}\frac{\underline{z}}{m^2},\\
&\lim_{\alpha\to 0}\frac{dd^*}{dm} = -\frac{F_{m0}-F_{y0}\frac{\underline{z}}{m^2}}{F_{d0}}.
\end{aligned} \tag{C17}$$

Now define the function obtained by evaluating (19) at the $\alpha\to 0$ entitlement level $y^E=\underline{z}/m$ and at $\alpha=0$,

$$\widehat{F}_0(d;m)\equiv F\big(d,\tfrac{\underline{z}}{m};m,0\big)=0.$$

This is exactly the quadratic (C2) characterizing $d_0^*(m)$, whose root is $d_0^*(m)$. By the implicit function theorem, therefore,

$$\frac{dd_0^*}{dm}=-\frac{\widehat{F}_{0,m}}{\widehat{F}_{0,d}}\bigg|_{d=d_0^*(m)},\qquad \widehat{F}_{0,d}\equiv\frac{\partial\widehat{F}_0}{\partial d},\quad \widehat{F}_{0,m}\equiv\frac{\partial\widehat{F}_0}{\partial m}, \tag{C18}$$

where

$$\widehat{F}_{0,d}\big|_{d=d_0^*} = F_d\big(d_0^*, \tfrac{\underline{z}}{m}; m, 0\big) = F_{d0},$$

$$\widehat{F}_{0,m}\big|_{d=d_0^*} = F_m\big(d_0^*, \tfrac{\underline{z}}{m}; m, 0\big) + F_y\big(d_0^*, \tfrac{\underline{z}}{m}; m, 0\big) \cdot \frac{d}{dm}\Big(\frac{\underline{z}}{m}\Big) = F_{m0} - F_{y0}\frac{\underline{z}}{m^2}.$$

Substituting these into (C18) gives

$$\frac{dd_0^*}{dm} = -\frac{\widehat{F}_{0,m}}{\widehat{F}_{0,d}} = -\frac{F_{m0} - F_{y0}\dfrac{\underline{z}}{m^2}}{F_{d0}}.$$

Hence, by (C17),

$$\lim_{\alpha\to 0}\frac{dd^*}{dm} = \frac{dd_0^*}{dm}.$$

Furthermore, from (C10), (C15), and (C16),

$$\lim_{\alpha\to 0}\frac{dy^{E*}}{dm} = -0 \times \frac{dd_0^*}{dm} - \frac{\underline{z}}{m^2} = -\frac{\underline{z}}{m^2}.$$

□

Lemma 6. *Suppose Assumption 1 holds and z_1 is small enough that the borrowing shutdown level in the $\alpha \to 0$ limit, $\underline{m}(0)$, is defined. Suppose further that $\underline{m}(0)$ satisfies the following two conditions.*

$$\textit{(CL6a)}\quad p_z(1-\underline{m}(0))\frac{\underline{z}}{\underline{m}(0)}(-\ln \underline{m}(0)) \le 1,$$

$$\textit{(CL6b)}\quad z_1\big(\underline{m}(0) - p_z\underline{z}(\ln \underline{m}(0))^2\big) \le \underline{z}.$$

Then

$$\lim_{m\to\underline{m}(0)}\lim_{\alpha\to 0}\frac{d\delta^*}{dm} < 0.$$

Conditions (CL6a) and (CL6b) are identical to conditions (CP2a) and (CP2b) that Proposition 2 imposes on the borrowing shutdown level $\underline{m}(0)$.

Proof of Lemma 6.

This lemma concerns $d\delta^*/dm$ in the limit $m \to \underline{m}(0)$. We first derive the conditions under which the premises of Lemma 4, (CL4a) and (CL4b), are satisfied as $m \to \underline{m}(0)$.

Consider first (CL4a). Setting $d_0^*(\underline{m}(0)) = 0$ in (C3), $\underline{m}(0)$ satisfies

$$1 - p_z(1 - \underline{m}(0))\frac{\underline{z}}{\underline{m}(0)} = p_z(-\ln \underline{m}(0))z_1, \tag{C19}$$

which, using (C5), can be written $\varphi\big(\underline{m}(0)\big) = 0$. With this in mind, taking the limit $m \to \underline{m}(0)$ in condition (CL4a) gives

$$\text{(CL4a)}\ z_1 \leq z_1^{max}(\underline{m}(0)) \equiv \frac{1 - p_z(1 - \underline{m}(0)) \cdot \dfrac{\underline{z}}{\underline{m}(0)}}{p_z(-\ln \underline{m}(0))} = z_1,$$

so condition (CL4a) is always satisfied.

Next, take the limit $m \to \underline{m}(0)$ in condition (CL4b). Evaluating the left-hand side of (C7) with $d_0^*(\underline{m}(0)) = 0$ in mind, (CL4b) becomes

$$\text{(CL4b)}\quad \frac{\underline{m}(0)}{\underline{z}} - (1 - \underline{m}(0))p_z(-\ln \underline{m}(0)) \geq 0.$$

Multiplying both sides by $\underline{z}/\underline{m}(0)$ (> 0) and rearranging, this is $p_z(1 - \underline{m}(0))\dfrac{\underline{z}}{\underline{m}(0)}(-\ln \underline{m}(0)) \leq 1$, which is none other than condition (CL6a) of this lemma. Hence, under (CL6a), condition (CL4b) is satisfied.

The evaluations of (CL4a) and (CL4b) above are at the endpoint $m = \underline{m}(0)$, but (CL4a) holds at every point of the right neighborhood $m > \underline{m}(0)$ because $z_1^{max}(m)$ is increasing in m ((C5)), and (CL4b) by continuity. Under condition (CL6a), therefore, Lemma 4 applies at each m in the right neighborhood, and the $\alpha \to 0$ limit of the equilibrium converges to the corner solution

$$\lim_{\alpha \to 0} y^{E*}(\alpha, m) = \frac{\underline{z}}{m}, \qquad \lim_{\alpha \to 0} d^*(\alpha, m) = d_0^*(m) \tag{C20}$$

(in particular, $d_0^*(\underline{m}(0)) = 0$ at $m = \underline{m}(0)$).

We next compute $d\delta^*/dm$ in the limit $m \to \underline{m}(0)$ and $\alpha \to 0$. As shown in Step 4(ii), at each m in the right neighborhood of $\underline{m}(0)$ the equilibrium quantities in the $\alpha \to 0$ limit converge to the corner solution (C20). Substituting this into (15), the $\alpha \to 0$ limit of the default probability can be written as the function of m

$$\delta_0(m) \;\equiv\; \lim_{\alpha \to 0} \delta^*(\alpha, m) \;=\; p_z\left[d_0^*(m) + (1 - m)\frac{\underline{z}}{m}\right].$$

By Lemma 5, $\lim_{\alpha\to 0} d\delta^*/dm = d\delta_0(m)/dm$, so differentiating $\delta_0(m)$ with respect to m gives

$$\lim_{\alpha\to 0}\frac{d\delta^*}{dm} \;=\; \frac{d\delta_0(m)}{dm} \;=\; p_z\left[\frac{dd_0^*}{dm} - \frac{\underline{z}}{m^2}\right].$$

Finally, taking the limit $m \to \underline{m}(0)$,

$$\lim_{m\to\underline{m}(0)}\lim_{\alpha\to 0}\frac{d\delta^*}{dm} \;=\; p_z\left[\left.\frac{dd_0^*}{dm}\right|_{m=\underline{m}(0)} - \frac{\underline{z}}{\underline{m}(0)^2}\right]. \tag{C21}$$

Now differentiate the quadratic (C2) defining $d_0^*(m)$ implicitly with respect to m and evaluate at $m = \underline{m}(0)$. Using $d_0^*(\underline{m}(0)) = 0$ and (C19),

$$\left.\frac{dd_0^*}{dm}\right|_{m=\underline{m}(0)} = \frac{\dfrac{\underline{z}}{\underline{m}(0)^2} + \dfrac{z_1}{\underline{m}(0)}}{2 + p_z z_1(\ln \underline{m}(0))^2}. \tag{C22}$$

Finally, substituting (C22) into (C21) gives

$$\lim_{m\to\underline{m}(0)}\lim_{\alpha\to 0}\frac{d\delta^*}{dm} = \frac{p_z}{\underline{m}(0)^2}\cdot\frac{z_1\big(\underline{m}(0) - p_z\underline{z}(\ln\underline{m}(0))^2\big) - \underline{z}}{2 + p_z z_1(\ln\underline{m}(0))^2}.$$

Hence

$$\lim_{m\to\underline{m}(0)}\lim_{\alpha\to 0}\frac{d\delta^*}{dm} < 0 \iff z_1\big(\underline{m}(0) - p_z\underline{z}(\ln\underline{m}(0))^2\big) < \underline{z}.$$

The inequality on the right is none other than condition (CL6b) of this lemma. Therefore, under (CL6a) and (CL6b), $\lim_{m\to\underline{m}(0)}\lim_{\alpha\to 0} d\delta^*/dm < 0$ holds. $\square$

Proof of Proposition 2. Lemma 6 established that $\lim_{m\to\underline{m}(0)}\lim_{\alpha\to 0}\frac{d\delta^*}{dm} < 0$. Since $d\delta^*/dm$ is continuous in (α, m), the negativity of this limit implies that $d\delta^*/dm < 0$ is preserved in the region where α is sufficiently close to zero and m sufficiently close to $\underline{m}(0)$. That is, in this region the default probability δ^* is decreasing in m. This proves Proposition 2. $\blacksquare$

D Proof of Proposition 3

To apply Lemma 4, we first verify that as $m \to 1$ its premises (CL4a) and (CL4b) are satisfied for any $z_1 > 0$. For condition (CL4a), $\lim_{m\to 1} z_1^{max}(m) \to +\infty$, so for any fixed $z_1 > 0$ the condition holds for all m sufficiently close to 1. For condition (CL4b), the right-hand side diverges to $+\infty$ as $m \to 1$, while applying l'Hôpital's rule to (C3) gives $\lim_{m\to 1^{-1}} d_0^*(m) = 1/(2p_z)$, so the left-hand side stays bounded. Hence condition (CL4b) also holds for all m sufficiently close to 1.

Lemma 4 therefore applies at each m in the left neighborhood of $m = 1$, and the $\alpha \to 0$ limit of the equilibrium converges to the corner solution

$$\lim_{\alpha\to 0} y^{E*}(\alpha, m) = \frac{\underline{z}}{m}, \qquad \lim_{\alpha\to 0} d^*(\alpha, m) = d_0^*(m).$$

We record the values of these limits as $m \to 1$. For entitlement setting, $\lim_{\alpha\to 0} y^{E*} = \underline{z}/m$ implies

$$\lim_{m\to 1}\lim_{\alpha\to 0} y^{E*} = \underline{z}. \tag{D1}$$

For debt issuance, as noted above, applying l'Hôpital's rule to (C3) yields

$$\lim_{m\to 1}\lim_{\alpha\to 0} d^* = d_0^*(1) = \frac{1}{2p_z}. \tag{D2}$$

Lemma 7. *Suppose Assumption 1 and the following condition hold.*

$$\textit{(CL7)} \quad z_1 > \frac{5\underline{z} - \overline{z}}{4}.$$

Then

$$\lim_{m\to 1}\lim_{\alpha\to 0} \frac{d\delta^*}{dm} > 0.$$

Condition (CL7) is identical to condition (CP3) imposed by Proposition 3.

Proof of Lemma 7.

We compute $d\delta^*/dm$ in the limit $m \to 1$ and $\alpha \to 0$. As in the proof of Proposition 2, substituting the corner solution into (14), the $\alpha \to 0$ limit of the default probability can be written as the function of m

$$\delta_0(m) \equiv \lim_{\alpha\to 0} \delta^*(\alpha, m) = p_z \left[d_0^*(m) + (1-m)\frac{\underline{z}}{m} \right].$$

By Lemma 5, $\lim_{\alpha\to 0} d\delta^*/dm = d\delta_0(m)/dm$; noting $\dfrac{d}{dm}\left[(1-m)\dfrac{\underline{z}}{m}\right] = -\dfrac{\underline{z}}{m^2}$, differentiating $\delta_0(m)$

and taking the limit $m \to 1$ gives

$$\lim_{m\to 1}\lim_{\alpha\to 0}\frac{d\delta^*}{dm} = \lim_{m\to 1} p_z\left[\frac{dd_0^*}{dm} - \frac{\underline{z}}{m^2}\right] = p_z\left[\left.\frac{dd_0^*}{dm}\right|_{m=1} - \underline{z}\right]. \tag{D3}$$

Applying the implicit function theorem to (C2) and noting $p_z = 1/(\overline{z} - \underline{z})$ gives

$$\left.\frac{dd_0^*}{dm}\right|_{m=1} = \frac{\overline{z} + 3\underline{z} + 4z_1}{8}. \tag{D4}$$

Finally, substituting (D4) into (D3) gives

$$\lim_{m\to 1}\lim_{\alpha\to 0}\frac{d\delta^*}{dm} = p_z \cdot \frac{\overline{z} - 5\underline{z} + 4z_1}{8}.$$

Hence

$$\lim_{m\to 1}\lim_{\alpha\to 0}\frac{d\delta^*}{dm} > 0 \iff z_1 > \frac{5\underline{z} - \overline{z}}{4}.$$

The inequality on the right is none other than condition (CL7) of this lemma. Therefore, under (CL7), $\lim_{m\to 1}\lim_{\alpha\to 0} d\delta^*/dm > 0$ holds. □

Proof of Proposition 3. Lemma 7 established that $\lim_{m\to 1}\lim_{\alpha\to 0}\frac{d\delta^*}{dm} > 0$. Since $d\delta^*/dm$ is continuous in (α, m), the positivity of this limit implies that $d\delta^*/dm > 0$ is preserved in the region where α is sufficiently close to zero and m sufficiently close to 1. That is, in a neighborhood of full protection $m = 1$, the default probability δ^* is increasing in m. This proves Proposition 3. ■

E Proof of Proposition 4

This Appendix analyzes the equilibrium in the limit $\alpha \to 0$ and examines, near the borrowing shutdown level $m \to \underline{m}(0)$, the relationship between the strength of entitlement protection m and the welfare of the two generations, $V_I^*(m)$ and $V_S^*(m)$, when α is sufficiently small.

The following lemma holds.

Lemma 8. *Suppose Assumption 1 and conditions (CL4a) and (CL4b) of Lemma 4 hold. Then, in* *the $\alpha \to 0$ limit, the marginal welfare effect for the issuing generation is*

$$\lim_{\alpha\to 0}\frac{dV_I^*}{dm} = \frac{p_z \underline{z}}{m^2}\left[\frac{d_0^*(m)}{z_1 + q(d_0^*(m), \underline{z}/m)d_0^*(m)} - \ln m\right] + \frac{p_z[d_0^*(m) + (1-m)\underline{z}/m] - 1}{m}. \tag{E1}$$

Moreover, at any m with $\lim_{\alpha\to 0} d\delta^*/dm \neq 0$, for sufficiently small α the marginal welfare effect for the successor generation satisfies

$$\operatorname{sign}\frac{dV_S^*}{dm} = -\operatorname{sign}\frac{d\delta^*}{dm}. \tag{E2}$$

Proof of Lemma 8.

The marginal effect on V_I^*

Since conditions (CL4a) and (CL4b) of Lemma 4 hold, note that by Lemma 4, $\lim_{\alpha\to 0} d^*(m) = d_0^*(m)$ and $\lim_{\alpha\to 0} y^{E*}(m) = \underline{z}/m$. Hence, from (28),

$$\begin{aligned}\lim_{\alpha\to 0}\frac{dV_I^*}{dm} =& \frac{p_z \frac{\underline{z}}{m} d_0^*(m)}{z_1 + q(d_0^*(m), \underline{z}/m)d_0^*(m)} - p_z\frac{\underline{z}}{m}\ln m + \frac{p_z\left[d_0^*(m) + (1-m)\frac{\underline{z}}{m}\right]}{m} \\ &+ \lim_{\alpha\to 0}\left(-\frac{\alpha\delta y^E}{X_d}\right).\end{aligned} \tag{E3}$$

By (C14), derived in the proof of Lemma 5, $\lim_{\alpha\to 0}(\alpha\delta m/X_d) = a_0(m)$, so using also $y^E \to \underline{z}/m$,

$$\lim_{\alpha\to 0}\left(-\frac{\alpha\delta y^E}{X_d}\right) = \lim_{\alpha\to 0}\left(-\frac{y^E}{m}\cdot\frac{\alpha\delta m}{X_d}\right) = -\frac{1}{m} + \frac{(1-m)p_z\underline{z}}{m^2}\left[\frac{d_0^*}{z_1 + q(d_0^*, \underline{z}/m)d_0^*} - \ln m\right].$$

Substituting this into (E3) and collecting the four terms yields (E1).

The marginal effect on V_S^*

Since the successor generation does not optimize, the marginal effect on its welfare is, from (29),

$$\frac{dV_S^*}{dm} = (1+\alpha)\left[\frac{d\delta^*}{dm}\ln X_d + \delta\frac{1}{X_d}\frac{dX_d}{dm} + p_z\left(\frac{dX_r}{dm}\ln X_r - \frac{dX_d}{dm}\ln X_d\right)\right]. \tag{E4}$$

We consider the limits of each term as $\alpha \to 0$.

First, from Lemma 4 and Lemma 5,

$$\lim_{\alpha\to 0}\frac{d\delta^*}{dm}\ln X_d = \frac{d\delta_0(m)}{dm}\cdot\lim_{\alpha\to 0}\ln X_d = -\infty,$$

$$\lim_{\alpha\to 0}\frac{dX_r}{dm}\ln X_r = \left(\frac{\underline{z}}{m^2} - \frac{dd_0^*(m)}{dm}\right)\ln\left(\overline{z} - d_0^*(m) - \frac{\underline{z}}{m}\right).$$

That is, the first term diverges while the third converges.

Consider next $\delta(1/X_d)(dX_d/dm)$. To shorten notation, define

$$\Psi \equiv \frac{\alpha\delta m}{X_d}, \qquad \Gamma \equiv \frac{d}{dm}\ln\Psi, \qquad \Phi \equiv \frac{d}{dm}\ln(\delta m).$$

The point is that while both $1/X_d$ and $\ln X_d$ diverge, Φ and Γ both converge to finite values.

(a) Decomposition. By the definition of Ψ, $X_d = \alpha\delta m/\Psi$, that is, $\ln X_d = \ln\alpha + \ln(\delta m) - \ln\Psi$. Since $\ln\alpha$ does not depend on m, differentiating both sides with respect to m gives

$$\frac{1}{X_d}\frac{dX_d}{dm} = \frac{d}{dm}\ln X_d = \Phi - \Gamma. \tag{E5}$$

We derive the limits of Φ and Γ in turn.

(b) Limits of Φ and related quantities. For Φ, from Lemma 4 and Lemma 5,

$$\lim_{\alpha\to 0}\Phi = \lim_{\alpha\to 0}\left[\frac{1}{\delta}\frac{d\delta^*}{dm} + \frac{1}{m}\right] = \frac{1}{\delta_0(m)}\frac{d\delta_0(m)}{dm} + \frac{1}{m}. \tag{E6}$$

By the same two Lemmas, the logarithmic derivative of $X_r = \overline{z} - d - y^E$, $\dfrac{1}{X_r}\dfrac{dX_r}{dm}$, also converges to a finite value, because $dX_r/dm = -dd^*/dm - dy^{E*}/dm$ converges and $X_r \to \overline{z} - d_0^*(m) - \underline{z}/m > 0$.

We also record, for use in (c) below, the limit of $\dfrac{da(\alpha)}{dm}$. From (C11),

$$a(\alpha) = -\frac{(1+\alpha)(1-m)p_z d}{z_1 + qd} + (1-m)p_z\ln m + \frac{1}{y^E}.$$

This is a smooth function of $\{d, y^E, m\}$, and its total derivative along the equilibrium is, by the chain rule,

$$\frac{da(\alpha)}{dm} = \frac{\partial a}{\partial d}\frac{dd^*}{dm} + \frac{\partial a}{\partial y^E}\frac{dy^{E*}}{dm} + \frac{\partial a}{\partial m}.$$

Write a at $\alpha = 0$ as $a_0(d, y^E, m) = a(d, y^E, m; 0)$. The partial derivatives $\partial a/\partial d$, $\partial a/\partial y^E$, and $\partial a/\partial m$ are continuous at the corner $(d, y^E) = (d_0^*(m), \underline{z}/m)$, so as $\alpha \to 0$, with the equilibrium $(d^*, y^{E*}) \to (d_0^*(m), \underline{z}/m)$, they converge to the corresponding partial derivatives of a_0. Moreover, by

Lemma 5, $dd^*/dm \to dd_0^*(m)/dm$ and $dy^{E*}/dm \to -\underline{z}/m^2$. Hence

$$\lim_{\alpha\to 0}\frac{da(\alpha)}{dm} = \frac{\partial a_0}{\partial d}\frac{dd_0^*(m)}{dm} + \frac{\partial a_0}{\partial y^E}\Big(-\frac{\underline{z}}{m^2}\Big) + \frac{\partial a_0}{\partial m} = \frac{da_0(m)}{dm},$$

where the last equality follows from the chain rule applied to $a_0(m) = a_0(d_0^*(m), \underline{z}/m, m)$.

(c) The limit of Γ. (C11) can be written $\Psi = a(\alpha) + \alpha p_z \ln(X_d/X_r)$. Differentiating both sides with respect to m gives

$$\frac{d\Psi}{dm} = \frac{da(\alpha)}{dm} + \alpha p_z\Big(\frac{1}{X_d}\frac{dX_d}{dm} - \frac{1}{X_r}\frac{dX_r}{dm}\Big).$$

On the left-hand side, $d\Psi/dm = \Psi\Gamma$ by the definition of Γ (a logarithmic derivative), and on the right-hand side, (E5) can be used for $\dfrac{1}{X_d}\dfrac{dX_d}{dm}$, so

$$\Psi\Gamma = \frac{da(\alpha)}{dm} + \alpha p_z\Big(\Phi - \Gamma - \frac{1}{X_r}\frac{dX_r}{dm}\Big).$$

Collecting the terms involving Γ on the left-hand side,

$$\big(\Psi + \alpha p_z\big)\Gamma = \frac{da(\alpha)}{dm} + \alpha p_z\Phi - \alpha p_z\frac{1}{X_r}\frac{dX_r}{dm}.$$

Now let $\alpha \to 0$. By (C14), the coefficient on the left-hand side satisfies $\Psi + \alpha p_z \to a_0(m) > 0$. On the right-hand side, by (b), Φ and $\dfrac{1}{X_r}\dfrac{dX_r}{dm}$ both converge to finite values, so the two terms multiplied by αp_z converge to 0, and the remaining $da(\alpha)/dm$ converges to $da_0(m)/dm$. Hence

$$\lim_{\alpha\to 0}\Gamma = \frac{1}{a_0(m)}\cdot\frac{da_0(m)}{dm}. \tag{E7}$$

(d) Combination. Substituting (E6) and (E7) into (E5) gives

$$\lim_{\alpha\to 0}\frac{1}{X_d}\frac{dX_d}{dm} = \frac{1}{\delta_0(m)}\frac{d\delta_0(m)}{dm} + \frac{1}{m} - \frac{1}{a_0(m)}\frac{da_0(m)}{dm}. \tag{E8}$$

Multiplying by $\delta \to \delta_0(m)$,

$$\begin{aligned}\lim_{\alpha\to 0}\delta\frac{1}{X_d}\frac{dX_d}{dm} &= \Big(\lim_{\alpha\to 0}\delta\Big)\Big(\lim_{\alpha\to 0}\frac{1}{X_d}\frac{dX_d}{dm}\Big)\\ &= \delta_0(m)\left(\frac{1}{\delta_0(m)}\frac{d\delta_0(m)}{dm} + \frac{1}{m} - \frac{1}{a_0(m)}\frac{da_0(m)}{dm}\right)\end{aligned}$$

$$= \frac{d\delta_0(m)}{dm} + \frac{\delta_0(m)}{m} - \frac{\delta_0(m)}{a_0(m)} \frac{da_0(m)}{dm}.$$

That is, the first term of the public goods channel, $\delta \frac{1}{X_d} \frac{dX_d}{dm}$, converges to a finite value as $\alpha \to 0$.

Finally, consider the remaining term of the public goods channel, $\frac{dX_d}{dm} \ln X_d$. Although $\ln X_d \to -\infty$ diverges, multiplying and dividing by X_d,

$$\frac{dX_d}{dm} \ln X_d = (X_d \ln X_d) \cdot \frac{1}{X_d} \frac{dX_d}{dm},$$

the standard limit $X_d \ln X_d \to 0$ as $X_d \to 0$ and (E8) yield

$$\begin{aligned} \lim_{\alpha \to 0} \left(\frac{dX_d}{dm} \ln X_d \right) &= \left(\lim_{\alpha \to 0} X_d \ln X_d \right) \left(\lim_{\alpha \to 0} \frac{1}{X_d} \frac{dX_d}{dm} \right) \\ &= 0 \cdot \left(\frac{1}{\delta_0(m)} \frac{d\delta_0(m)}{dm} + \frac{1}{m} - \frac{1}{a_0(m)} \frac{da_0(m)}{dm} \right) \\ &= 0. \end{aligned}$$

In sum, as $\alpha \to 0$, the second, third, and fourth terms on the right-hand side of (E4) all converge to finite values while the first diverges to $-\infty$. Hence, for α sufficiently close to zero, the divergent first term dominates the other, convergent terms in absolute value, and the sign of $\frac{dV_S^*}{dm}$ coincides with that of the first term. For α sufficiently close to zero, X_d is also close to zero and $\ln X_d < 0$, so

$$\operatorname{sign} \frac{dV_S^*}{dm} = \operatorname{sign} \left(\frac{d\delta^*}{dm} \ln X_d \right) = -\operatorname{sign} \frac{d\delta^*}{dm},$$

which is none other than (E2). □

Proof of Proposition 4. Finally, using Lemma 8, we evaluate (E1) and (E2) at $m \to \underline{m}(0)$.

First, evaluate (E1) at $m = \underline{m}(0)$. Noting that $d_0^*(\underline{m}(0)) = 0$ by the definition of $\underline{m}(0)$, and using (C19),

$$\begin{aligned} \lim_{\alpha \to 0} \frac{dV_I^*}{dm} \bigg|_{m=\underline{m}(0)} &= \frac{p_z \underline{z}}{\underline{m}(0)^2} (-\ln \underline{m}(0)) + \frac{p_z (1-m) \frac{\underline{z}}{\underline{m}(0)} - 1}{\underline{m}(0)} \\ &= \frac{p_z (-\ln \underline{m}(0))}{\underline{m}(0)} \left(\frac{\underline{z}}{\underline{m}(0)} - z_1 \right). \end{aligned}$$

That is,

$$\lim_{\alpha \to 0} \left. \frac{dV_I^*}{dm} \right|_{m=\underline{m}(0)} \gtreqless 0 \iff z_1 \underline{m}(0) \lesseqgtr \underline{z}. \tag{E9}$$

Hence, when z_1 is sufficiently small,

$$\lim_{\alpha \to 0} \left. \frac{dV_I^*}{dm} \right|_{m=\underline{m}(0)} > 0.$$

Next, evaluate (E2) at $m = \underline{m}(0)$. When conditions (CP2a) and (CP2b) assumed by the proposition hold, conditions (CL6a) and (CL6b) of Lemma 6 also hold, so Lemma 6 applies directly and $\lim_{m \to \underline{m}(0)} \lim_{\alpha \to 0} d\delta^*/dm < 0$. By the sign relation (E2), therefore, in a sufficiently close right neighborhood of $\underline{m}(0)$ and for sufficiently small α,

$$\lim_{\alpha \to 0} \left. \frac{dV_S^*}{dm} \right|_{m=\underline{m}(0)} > 0.$$

Since dV_I^*/dm and dV_S^*/dm are both continuous in (α, m), the positivity of these values implies that $dV_I^*/dm > 0$ and $dV_S^*/dm > 0$ hold simultaneously in the region where α is sufficiently close to zero and m sufficiently close to $\underline{m}(0)$. That is, in this region a marginal strengthening of entitlement protection (a rise in m) raises the welfare of both generations simultaneously, delivering a Pareto improvement. This proves Proposition 4. ■

F Proof of Proposition 5

Proposition 5 asserts that near full protection, $m \to 1$, both $dV_I^*/dm < 0$ and $dV_S^*/dm < 0$ hold. The conditions are shared with Proposition 3 (Assumption 1 and condition (CP3)). As shown in Appendix D, as $m \to 1$ conditions (CL4a) and (CL4b) of Lemma 4 hold automatically for any $z_1 > 0$. Lemma 8 therefore applies directly in the left neighborhood of $m = 1$, and condition (CP2a) required by Proposition 2 is unnecessary. The proof consists of evaluating, at this endpoint $m = 1$, the same Lemma 8 used in the proof of Proposition 4—the $\alpha \to 0$ forms of the marginal welfare effects, (E1) and (E2).

Proof of Proposition 5. First, evaluate (E1) at $m = 1$. By (D1) and (D2) in Appendix D, as $m \to 1$,

$$y_0^{E*} \to \underline{z}, \quad d_0^*(1) = \frac{1}{2p_z}, \quad \delta_0(1) = p_z d_0^*(1) = \frac{1}{2}, \quad q(d_0^*(1), \underline{z}) = \frac{1}{2}, \quad \ln m \to 0$$

so (E1) and (CP3) give

$$\lim_{\alpha \to 0} \left. \frac{dV_I^*}{dm} \right|_{m=1} = \frac{1}{2} \left[\frac{\underline{z}}{z_1 + \frac{\overline{z} - \underline{z}}{4}} - 1 \right] < 0.$$

Next, evaluate (E2) at $m = 1$. When condition (CP3) assumed by the proposition holds, condition (CL7) of Lemma 7 also holds (the two are the same expression), so Lemma 7 applies directly and $\lim_{m \to 1} \lim_{\alpha \to 0} d\delta^*/dm > 0$. By the sign relation (E2), therefore, in a sufficiently close left neighborhood of 1 and for sufficiently small α,

$$\lim_{\alpha \to 0} \left. \frac{dV_S^*}{dm} \right|_{m=1} < 0.$$

Since dV_I^*/dm and dV_S^*/dm are both continuous in (α, m), the negativity of these values implies that $dV_I^*/dm < 0$ and $dV_S^*/dm < 0$ hold simultaneously in the region where α is sufficiently close to zero and m sufficiently close to 1. That is, in this region a marginal relaxation of entitlement protection (a fall in m) raises the welfare of both generations simultaneously, delivering a Pareto improvement. This proves Proposition 5. ■

G The procedure for delineating the parameter range and details of the equilibrium quantities

Delineating the parameter range

Figure 8 in Section 7 shows the range in the (z_1, α) plane over which all the claims of Propositions 2–5 hold. This Appendix gives the conditions and the procedure used for that determination. The parameters $\underline{z} = 1$ and $\overline{z} = 6.5$ are kept fixed at their baseline values.

Table 1 first collects the conditions imposed by Propositions 1–5. All of them, including the parts stated in the form "z_1 is sufficiently small," are written below as explicit conditions on the exogenous parameters $(\underline{z}, \overline{z}, z_1, \alpha)$; the determination follows these.

First, define the borrowing shutdown level in the $\alpha \to 0$ limit, $\underline{m}(0)$. The first-order condition at

	Source	Condition
(C1)	Proposition 1, Eq. (G2)	$z_1 < (1+\alpha)y_1^E$ (sufficient condition)
(C2)	(CP2a)	$p_z(1-\underline{m}(0))\dfrac{\underline{z}}{\underline{m}(0)}\big(-\ln \underline{m}(0)\big) \leq 1$
(C3)	(CP2b)	$z_1\big(\underline{m}(0) - p_z\underline{z}(\ln \underline{m}(0))^2\big) \leq \underline{z}$
(C4)	(CP3)	$z_1 > \dfrac{5\underline{z}-\overline{z}}{4}$
(C5)	Eq. (E9)	$z_1\underline{m}(0) < \underline{z}$
(C6)	each Proposition	α sufficiently small (no explicit form given)
Proposition 2: (C1)(C2)(C3)(C6)		Proposition 4: (C1)(C2)(C3)(C5)(C6)
Propositions 3 and 5: (C1)(C4)(C6)		

Table 1: The parameter conditions imposed by Propositions 1–5.

$d = 0$ is given by (B6). Noting that $y^E(0) \to \underline{z}/m$ as $\alpha \to 0$, applying $\alpha \to 0$ to that equation gives

$$\frac{1 - p_z(1-m)\underline{z}/m}{z_1} + p_z \ln m = 0. \tag{G1}$$

The level $\underline{m}(0)$ is the root of (G1); it is determined by $(\underline{z}, \overline{z}, z_1)$ alone and does not depend on α.

Next, we derive condition (C1). The proof of Proposition 1 uses $\lim_{m\to1} \widehat{F}(0) > 0$ as the condition for the borrowing shutdown level to exist. Letting $m \to 1$ in (B7), $y_1^E \equiv \lim_{m\to1} y^E(0)$ is defined as the root of

$$\frac{1}{y_1^E} = \alpha p_z \ln \frac{\overline{z} - y_1^E}{\underline{z} - y_1^E}.$$

With this in mind, letting $m \to 1$ in (B6), since $(1-m)y^E(0) \to 0$ and $p_z \ln m \to 0$,

$$\lim_{m\to1} \widehat{F}(0) = \frac{1+\alpha}{z_1} + \alpha p_z \ln \frac{\underline{z} - y_1^E}{\overline{z} - y_1^E} = \frac{1+\alpha}{z_1} - \frac{1}{y_1^E}.$$

Hence $\lim_{m\to1} \widehat{F}(0) > 0$ is equivalent to

$$z_1 < (1+\alpha)y_1^E, \tag{G2}$$

which is (C1). Whereas (G1) is the $\alpha \to 0$ limit of (B6), (C1) is obtained from the $m \to 1$ limit of the same equation.

Conditions (C2)–(C5) in Table 1 are determined solely by $(\underline{z}, \overline{z}, z_1)$ through $\underline{m}(0)$ and do not involve α; all are satisfied at the baseline.

Conditions (C1) and (C6) are handled differently. (C1) is a sufficient, not a necessary, condition

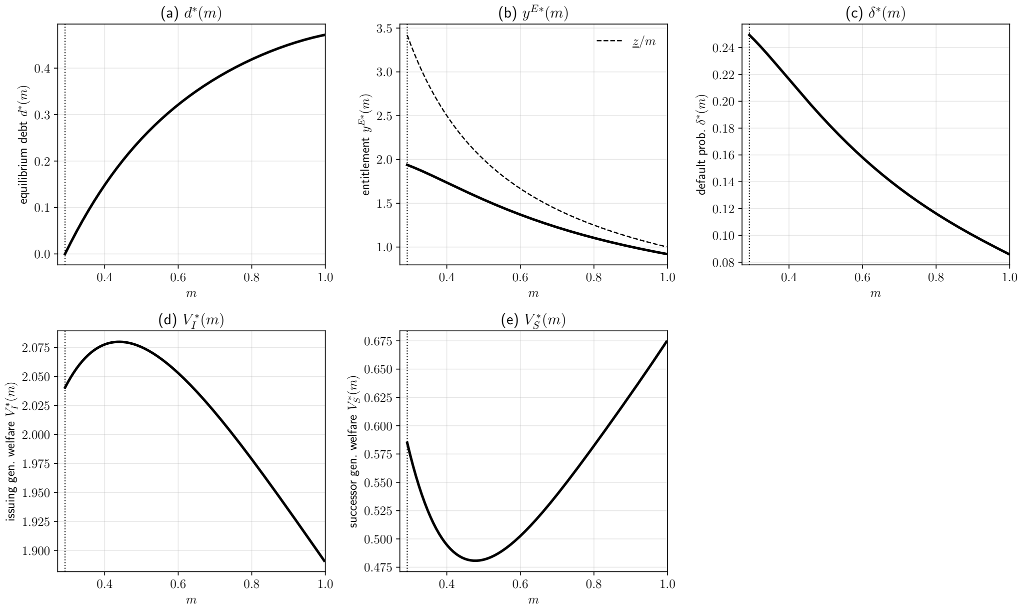

Figure 10: Equilibrium quantities in the case where only α is raised from the baseline ($\underline{z} = 1$, $\overline{z} = 6.5$, $z_1 = 2.5$, $\alpha = 0.6$). The vertical dotted line marks the borrowing shutdown level $\underline{m} \simeq 0.292$. Panel (a): $d^*(m)$. Panel (b): $y^{E*}(m)$ (the dashed line is the level $\underline{z}/m$ that exhausts the resources available in default). Panel (c): $\delta^*(m)$. Panel (d): $V_I^*(m)$. Panel (e): $V_S^*(m)$. Both welfare measures omit constant terms.

for the existence of equilibrium (see the remark in Appendix B), and equilibria may exist outside the range where it holds. Indeed, the baseline in the text does not satisfy (C1), yet an interior equilibrium exists over all of $m \in [\underline{m}, 1]$. We therefore verify the existence of an interior equilibrium directly at each point, without relying on (C1): after computing $\underline{m}$, we partition the interval $[\underline{m},\ 1]$ and check that at every m the equilibrium (d^*, y^{E*}) belongs to the feasible set Ω and satisfies $0 < \delta^* < 1$. If even one point fails, the case is classified as "no interior equilibrium."

(C6) has no explicit form. The purpose of this Appendix is to delineate numerically how large an α this condition actually permits. At points where an interior equilibrium exists, we numerically differentiate the slopes of δ^*, V_I^*, and V_S^* with respect to m near the two endpoints and determine whether the claims of the four propositions hold. The result of this determination is Figure 8 in the text.

Details of the equilibrium quantities

For the two cases treated in Section 7—raising only α from the baseline (Figure 10) and raising only z_1 (Figure 11)—the equilibrium quantities $(d^*, y^{E*}, \delta^*, V_I^*, V_S^*)$ are shown as functions of m.

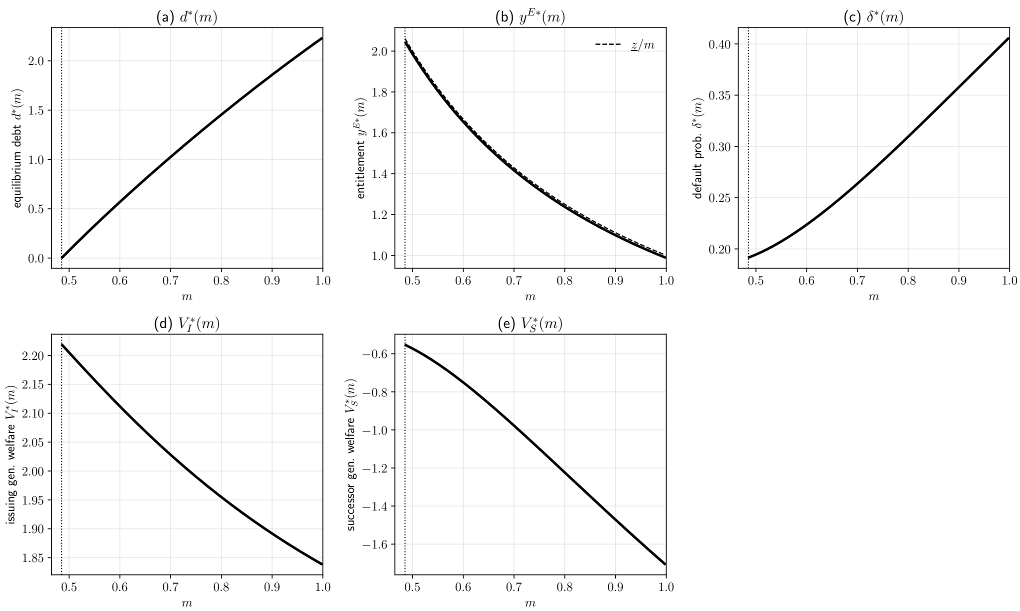

Figure 11: Equilibrium quantities in the case where only z_1 is raised from the baseline ($\underline{z} = 1,\ \overline{z} = 6.5,\ z_1 = 5.0,\ \alpha = 0.03$). The vertical dotted line marks the borrowing shutdown level $\underline{m} \simeq 0.485$. Panel (a): $d^*(m)$. Panel (b): $y^{E*}(m)$ (the dashed line is the level $\underline{z}/m$ that exhausts the resources available in default). Panel (c): $\delta^*(m)$. Panel (d): $V_I^*(m)$. Panel (e): $V_S^*(m)$. Both welfare measures omit constant terms.

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